 HW4 Problem:
An economist derives the following data regarding the supply and demand for coffee.

\[ Q_d = \text{demand for coffee} \]
\[ Q_s = \text{supply for coffee} \]
\[ P = \text{price for coffee} \]
\[ Q_d = 15 - \frac{p^2}{15} \]
\[ Q_s = 3 + 0.808P \]

The equilibrium price and quantity are not known but are denoted by \( P^* \) and \( Q^* \).

a) Sketch the curves on a graph and identify the equilibrium price and quantity \( (P^* \text{ and } Q^*) \).

b) Using a definite integral calculate the consumer surplus

c) Using a definite integral calculate the producer surplus

Solution:

a) 
\[ Q_d = Q_s \]
\[ 15 - \frac{p^2}{15} = 3 + 0.808P \]

This can be rearranged to read:

\[ f(P) = \frac{p^2}{15} + 0.808P - 12 = 0 \]
\[ p^2 + 12.12P - 180 = 0 \]

In this case \( a = 1, b = 12.12, \text{ and } c = -180 \)

Using the quadratic formula you can calculate:

\[ p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12.12 \pm \sqrt{(12.12)^2 - 4(1)(-180)}}{2(1)} = \frac{-12.12 \pm \sqrt{146.89 + 720}}{2} \]
\[ P_{1,2} = \frac{-12.12 \pm \sqrt{866.89}}{2} = \frac{-12.12 \pm 29.44}{2} = 8.66 \text{ or } -20.78 \]

We disregard second root as price cannot be negative and we conclude that the market equilibrium is achieved when the price is £8.66 per unit. By plugging \( P^* = 8.66 \) into either the demand or supply equation (which are equal at the market equilibrium) we can also find out how much is bought and sold at this price.

\[ Q^* = 3 + 0.808P^* = 3 + 0.808(8.66) = 10 \text{ units} \]
\[ Q^* = 15 - \frac{(8.66)^2}{15} = 10 \text{ units} \]
The supply and demand curves, and the market equilibrium, are illustrated on the graph below on the next page.

b) **Calculation of consumer surplus**
Consumer surplus is shown by the area $A$.

\[
\text{Consumer surplus} = \int_0^{Q^*} f(Q_d) dQ_d - P^* Q^*
\]

We must rearrange the demand function in terms of $P = f(Q_d)$ to get:

\[
\begin{align*}
\frac{P^2}{15} &= 15 - Q_d \\
\frac{P^2}{225} &= 225 - 15Q_d \\
P &= \sqrt{225 - 15Q_d}
\end{align*}
\]

\[
\text{Area } A = \int_0^{10} (\sqrt{225 - 15Q_d}) dQ_d - (8.66) (10)
\]

Change the variable as follows to integrate demand function:

\[
\begin{align*}
u &= (225 - 15Q_d) \\
\int u^{(\frac{3}{2})} du &= \frac{u^{(\frac{3}{2})+1}}{(\frac{3}{2})+1} + C \\
du &= -15dQ_d
\end{align*}
\]

\[
\text{Area } A = \int_0^{10} \left[ \frac{(225 - 15Q_d)^{\frac{3}{2}+1}}{\frac{3}{2}+1} \right]^{10}_0 - 86.6
\]

\[
\text{Area } A = \frac{-1}{15} \left[ \frac{(225 - 15Q_d)^{(3/2)}}{(3/2)} \right]^{10}_0 - 86.6
\]

\[
\text{Area } A = \frac{-1}{22.5} \left[ (225 - 15Q_d)^{\frac{3}{2}} \right]^{10}_0 - 86.6
\]

\[
\text{Area } A = \frac{-1}{22.5} \left[ (225 - 15(10))^1.5 \right] - (225)^1.5 - 86.6 = 34.53 \text{ units}
\]

c) **Calculation of producer surplus**
Producer surplus is shown by the area $B$.

\[
\text{Producer surplus} = P^* Q^* - \int_3^{Q^*} f(Q_s) dQ_s
\]

We must rearrange the supply function in terms of $P = f(Q_s)$ to get:

\[
\begin{align*}
Q_s &= 3 + 0.808P \\
P &= \frac{1}{0.808} Q_s - \frac{3}{0.808} = 1.238Q_s - 3.713
\end{align*}
\]

\[
\text{Area } B = (8.66)(10) - \int_3^{10} (1.238Q_s - 3.713) dQ_s
\]
\[ Area\ B = 86.6 - \left[ 1.238 \left( \frac{(Q_s)^2}{2} - 3.713Q_s \right) \right] ^{10} \]

\[ Area\ B = 86.6 - \left\{ \left[ 1.238 \left( \frac{(10)^2}{2} - 3.713(10) \right) \right] - \left[ 1.238 \left( \frac{(3)^2}{2} - 3.713(3) \right) \right] \right\} = 56.26\ \text{units} \]