1. (10 p) Evaluate the integral using integration by parts:
\[ \int (x + 1)^2 e^x \, dx \]

2. (10 p) Evaluate the integral of the following power product of the trigonometric functions:
\[ \int \cos^3 x \sin^4 x \, dx \]

3. (15 p) Assuming that the following parametric equations define \( x \) and \( y \) implicitly as differentiable functions \( x = f(t), y = g(t) \), find the slope of the curve \( x = f(t), y = g(t) \) at the given value of \( t = 0 \):
\[ x + 2x^{3/2} = t^2 + t \; ; \; y \sqrt{t^2 + 1 + 2t \sqrt{t}} = 4 \; ; \; t = 0 \]

4. (15 p) Find the length of the curve given by the polar coordinate equation as follows:
\[ r = 8 \sin^3 \left( \frac{\theta}{3} \right) ; \; 0 \leq \theta \leq \frac{\pi}{4} \]

Length of a polar curve:
\[ L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta \]

5. (10 p) Use the integral test to determine if the following series converges or diverges. Be sure to check that the conditions of the Integral Test are satisfied:
\[ \sum_{n=1}^{\infty} e^{-2n} \]

6. (10 p) Use any method to determine if the following series converges or diverges:
\[ \sum_{n=1}^{\infty} n! e^{-n} \]

7. (10 p) Find the Taylor polynomials of the orders 0, 1, 2, and 3 generated by \( f(x) \) at \( a \):
\[ f(x) = \frac{1}{x} ; \; a = 2 \]

8. (10 p) Find the Maclaurin series for the function \( f(x) = \frac{1}{1+x} \)

9. (15 p) An important partial differential equation that describes the fall and rise of the water \((w: \text{wave height})\) as the waves go by \((x: \text{distance})\) in time \(t\) can be represented by the one-dimensional wave equation as follows:
\[ \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \]

Show that \( w = f(u) \), where \( f \) is a differentiable function of \( u \) and \( u = a(x + ct) \), where \( a \) is a constant satisfies the wave equation given above where \( c \) is the velocity with which the waves are propagated.

10. a) (15 p) Express \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) as the following function of \( u \) and \( v \) both by using the Chain Rule and by expressing \( z \) directly in terms of \( u \) and \( v \) before differentiating:
\[ z = 4e^x \ln y \; ; \; x = \ln(u \cos v) \; ; \; y = u \sin v \]

b) (5 p) Evaluate \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \) at the given point \((u, v) = (1.3, \frac{\pi}{6})\).

Taylor’s Formula:
\[ f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^n(a)}{n!}(x - a)^n + R_n(x) \]
1. Apply IBP (Integration by parts) as follows:

\[ \int u\,dv = uv - \int v\,du \]

\[ u = (x + 1)^2, \, du = 2(x + 1)dx; \, dv = e^x\,dx, \, v = e^x \]

\[ l = \int (x + 1)^2 e^x\,dx = (x + 1)^2(e^x) - \int e^x(2x + 1)\,dx = (x + 1)^2(e^x) - 2 \int e^x(x + 1)\,dx \]

Apply again IBP for the inside integral called \( A = \int e^x(x + 1)\,dx \) as follows:

\[ u = (x + 1), \, du = dx; \, dv = e^x\,dx, \, v = e^x \]

\[ A = \int e^x(x + 1)\,dx = (x + 1)(e^x) - \int e^x\,dx = e^x(x + 1) - e^x + C' \]

\[ l = (x + 1)^2(e^x) - 2[e^x(x + 1) - e^x + C'] \]

\[ l = [(x + 1)^2 - 2(x + 1) + 2](e^x) + C \]

2.

\[ \int \cos^3 x \sin^4 x\,dx = \int \cos^2 x \sin^4 x \cos x\,dx \]

Apply u-substitution as follows:

\[ u = \sin x, \, du = \cos x\,dx \]

\[ \int \sin^4 x(1 - \sin^2 x)\cos x\,dx = \int u^4(1 - u^2)\,du = \int u^4\,du - \int u^6\,du = \frac{u^5}{5} - \frac{u^7}{7} + C \]

\[ \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C \]

3. Given: \( x + 2x^{3/2} = t^2 + t; \quad y\sqrt{t+1} + 2t\sqrt{y} = 4; \quad t = 0 \)

Implicit differentiation of these equations respectively gives:

\[ \frac{dx}{dt} + 3x^{1/2} \frac{dx}{dt} = 2t + 1 \rightarrow (1 + 3x^{1/2}) \frac{dx}{dt} = 2t + 1 \rightarrow \frac{dx}{dt} = \frac{2t + 1}{(1 + 3x^{1/2})} \]

\[ \frac{dy}{dt} \sqrt{t+1} + y \left(\frac{1}{2}\right) (t+1)^{-1/2} + 2\sqrt{y} + 2t \left(\frac{1}{2} y^{-1/2}\right) \frac{dy}{dt} = 0 \]

\[ \rightarrow \frac{dy}{dt} \sqrt{t+1} + \frac{y}{2\sqrt{t+1}} + 2\sqrt{y} + \left(\frac{t}{\sqrt{y}}\right) \frac{dy}{dt} = 0 \rightarrow \left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right) \frac{dy}{dt} = 0 \]

\[ = -\frac{y}{2\sqrt{t+1}} - 2\sqrt{y} \rightarrow \frac{dy}{dt} = \frac{-\frac{y}{2\sqrt{t+1}} - 2\sqrt{y}}{\sqrt{t+1} + \frac{t}{\sqrt{y}}} = \frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}} \]

\[ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}}{\left(\frac{2t + 1}{(1 + 3x^{1/2})}\right)}; \]

\[ t = 0 \rightarrow x + 2x^{3/2} = 0 \rightarrow x(1 + 2x^{1/2}) = 0; \rightarrow x = 0; \, t = 0 \]

\[ t = 0 \rightarrow y\sqrt{0+1} + 2(0)\sqrt{y} = 4 \rightarrow y = 4 \]
\[
\frac{dy}{dx} \bigg|_{t=0} = \left( \frac{-4\sqrt{4} - 4(4)\sqrt{0 + 1}}{2\sqrt{4}(0 + 1) + 2(0)\sqrt{0 + 1}} \right) = -6
\]

4. The length of the curve given by the polar coordinate equation as follows:

\[
r = 8 \sin^3\left(\frac{\theta}{3}\right); \quad 0 \leq \theta \leq \frac{\pi}{4} \quad \Rightarrow \quad \frac{dr}{d\theta} = 8 \sin^2\left(\frac{\theta}{3}\right) \cos\left(\frac{\theta}{3}\right)
\]

\[
L = \int_0^{\frac{\pi}{4}} \sqrt{8 \sin^3\left(\frac{\theta}{3}\right)^2 + \left[8 \sin^2\left(\frac{\theta}{3}\right) \cos\left(\frac{\theta}{3}\right)\right]^2} \, d\theta = \int_0^{\frac{\pi}{4}} \sqrt{64 \sin^4\left(\frac{\theta}{3}\right)} \, d\theta
\]

\[
= \int_0^{\frac{\pi}{4}} 8 \sin^2\left(\frac{\theta}{3}\right) \, d\theta = \int_0^{\frac{\pi}{4}} 8 \left[1 - \cos\left(\frac{2\theta}{3}\right)\right] \, d\theta = \int_0^{\frac{\pi}{4}} 4 - 4 \cos\left(\frac{2\theta}{3}\right) \, d\theta
\]

\[
= \left[4\theta - 6 \sin\left(\frac{2\theta}{3}\right)\right]_0^{\frac{\pi}{4}} = 4 \left(\frac{\pi}{4}\right) - 6 \sin\left(\frac{\pi}{6}\right) - 0 = \pi - 3
\]

5. \(f(x) = e^{-2x}\) is positive, continuous, and decreasing for \(x \geq 1\).

\[
\int_1^\infty e^{-2x} \, dx = \lim_{b \to \infty} \int_1^b e^{-2x} \, dx = \lim_{b \to \infty} \left[-\frac{1}{2} e^{-2x}\right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{2} e^{2b} + \frac{1}{2} e^2\right) = \frac{1}{2} e^2
\]

\[
\int_1^\infty e^{-2x} \, dx \text{ converges} \quad \sum_{n=1}^{\infty} e^{-2n} \text{ converges}
\]

6. The series given diverges by the Ratio Test as follows:

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n + 1)! e^{-(n+1)}}{n! e^{-n}} = \lim_{n \to \infty} \frac{(n + 1)! e^n}{e^{n+1} n!} = \lim_{n \to \infty} \frac{n + 1}{e} = \infty
\]

7. Taylor’s Formula:

\[
f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^n(a)}{n!}(x - a)^n + R_n(x)
\]

\[
R_n(x) = \frac{f^{n+1}(c)}{(n+1)!} (x-a)^{n+1} \text{ for some } c \text{ between } a \text{ and } x
\]

Apply Taylor’s formula given above to the function up to 3rd order derivative as follows:

\[
f(x) = \frac{1}{x} = -x^{-1}; \quad f(a) = f(2) = \frac{1}{2}; \quad f'(x) = -x^{-2} \Rightarrow f'(2) = -\frac{1}{4}; \quad f''(x) = 2x^{-3}
\]

\[
\Rightarrow f''(a) = f''(2) = \frac{1}{4}; \quad f'''(x) = -6x^{-4} \Rightarrow f'''(a) = f'''(2) = -\frac{3}{8}
\]

Zero, first, second and third order polynomials generated by \(f(x) = \frac{1}{x}\) according to Taylor’s formula were given respectively as follows:

\[
P_0(x) = f(a) = \frac{1}{2}; \quad P_1(x) = f(a) + f'(a)(x - a) = \frac{1}{2} - \frac{1}{4}(x - 2)
\]

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\[ P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 \]
\[ P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 = P_2(x) - \frac{1}{16}(x-2)^3 \]

8. Maclaurin series generated by \( f(x) = \frac{1}{1+x} \) is:
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots
\]
We need to find \( f(0), f'(0), f''(0), \ldots \) according to the formula given above as follows:
\[
f(x) = \frac{1}{1+x}; f'(x) = -(1 + x)^{-2}; f''(x) = 2(1 + x)^{-3}; f'''(x) = -3!(1 + x)^{-4} \rightarrow f^n(x)
\]
\[
= (-1)^n n! (1 + x)^{-n-1} \quad \Rightarrow \quad f(0) = 1; f'(0) = -1; f''(0) = 2; f'''(0) = -3!; \ldots
\]
\[
\rightarrow f^n(0) = (-1)^n n!
\]
\[
\sum_{n=0}^{\infty} (-1)^n n! x^n = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots
\]

9. To show the following equation given to be a solution of the partial differential equation, we need to take second order partial derivative according to \( t \) and second order partial derivative according to \( x \) as follows:
\[
w = f(u), \text{where } f \text{ is a differentiable function of } u \text{ and } u = a(x + ct)
\]
\[
\frac{\partial w}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} = \frac{\partial f}{\partial u} (ac)
\]
\[
\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} (a)
\]
\[
\frac{\partial^2 w}{\partial x^2} = \left( a \frac{\partial^2 f}{\partial u^2} \right) a = a^2 \frac{\partial^2 f}{\partial u^2}
\]
\[
\frac{\partial^2 w}{\partial t^2} = (ac) \frac{\partial^2 f}{\partial u^2} (ac) = a^2 c^2 \frac{\partial^2 f}{\partial u^2}
\]
\[
\text{Thus}
\]
\[
\frac{\partial^2 w}{\partial t^2} = a^2 c^2 \frac{\partial^2 f}{\partial u^2} = c^2 \frac{\partial^2 w}{\partial x^2}
\]

10. a) By using Chain Rule:
\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = (4e^{x\ln y}) \left( \frac{\cos v}{u \cos v} \right) + \left( \frac{4e^x}{y} \right) (\sin v) = \frac{4e^{x\ln y}}{u} + \frac{4e^x \sin v}{y}
\]
\[
= \frac{4(u \cos v) \ln(u \sin v)}{u} + \frac{4(u \cos v) (\sin v)}{u \sin v} = 4(\cos v) \ln(u \sin v) + 4(\cos v)
\]
\[\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = (4e^{x} \ln y) \left( \frac{-\sin v}{u \cos v} \right) + \left( \frac{4e^{x}}{y} \right) (u \cos v) = - (4e^{x} \ln y) (\tan v) + \frac{4e^{x} u \cos v}{y} \]

\[= - [4(u \cos v) \ln (u \sin v)] (\tan v) + \frac{4(u \cos v) (u \cos v)}{u \sin v} \]

\[= (-4u \sin v) \ln (u \sin v) + \frac{4u \cos^{2} v}{u \sin v} \]

By expressing \( z \) directly in terms of \( u \) and \( v \) before differentiating:

\[z = 4e^{x} \ln y = 4(u \cos v) \ln (u \sin v)\]

Differentiation of the last expression gives the same result as follows:

\[\frac{\partial z}{\partial u} = (4 \cos v) \ln (u \sin v) + 4(u \cos v) \left( \frac{\sin v}{u \sin v} \right) = (4 \cos v) \ln (u \sin v) + 4 \cos v ; \quad \frac{\partial z}{\partial v} = (-4u \sin v) \ln (u \sin v) + 4(u \cos v) \left( \frac{u \cos v}{u \sin v} \right) \]

\[= (-4u \sin v) \ln (u \sin v) + \frac{4u \cos^{2} v}{u \sin v} \]

b)

\[\left. \frac{\partial z}{\partial u} \right|_{(u,v)} = \left. \frac{\partial z}{\partial u} \right|_{(1.3, \pi/6)} = \left( 4 \cos \frac{\pi}{6} \right) \ln \left( 1.3 \sin \frac{\pi}{6} \right) + 4 \cos \frac{\pi}{6} = 1.97 \]

\[\left. \frac{\partial z}{\partial v} \right|_{(u,v)} = \left. \frac{\partial z}{\partial v} \right|_{(1.3, \pi/6)} = \left[ -4(1.3) \sin \frac{\pi}{6} \right] \ln \left( 1.3 \sin \frac{\pi}{6} \right) + \frac{4(1.3) \cos^{2} \frac{\pi}{6}}{\sin \frac{\pi}{6}} = 8.92 \]