1. (10 p) Evaluate the integral using integration by parts:
\[ \int e^x \cos 2x \, dx \]

2. (10 p) Evaluate the integral of the following power product of the trigonometric functions:
\[ \int \sin^5 x \cos^5 x \, dx \]

3. (15 p) Assuming that the following parametric equations define \( x \) and \( y \) implicitly as differentiable functions \( x = f(t), y = g(t) \), find the slope of the curve \( x = f(t), y = g(t) \) at the given value of \( t = 4 \):
\[ x = \sqrt{5 - \sqrt{t}} \quad ; \quad y(t - 1) = \sqrt{t} \quad ; \quad t = 4 \]

4. (20 p) Find the length and the area of the curve given by the polar coordinate equation as follows:
\[ r = 2 \sin \theta + 2 \cos \theta \quad ; \quad 0 \leq \theta \leq \frac{\pi}{2} \]
Length and area of a polar curve:
\[ L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta \]
\[ A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta \]

5. (10 p) Use the integral test to determine if the following series converges or diverges. Be sure to check that the conditions of the Integral Test are satisfied:
\[ \sum_{n=1}^{\infty} \frac{1}{n^4 + 4} \]

6. (10 p) Use any method to determine if the following series converges or diverges:
\[ \sum_{n=1}^{\infty} n^2 e^{-n} \]

7. (10 p) Find the Taylor polynomials of the orders 0, 1, 2, and 3 generated by \( f(x) \) at \( a \):
\[ f(x) = \ln(1 + x) \quad ; \quad a = 0 \]

8. (10 p) Find the Maclaurin series for the function \( f(x) = xe^x \)

9. (15 p) An important partial differential equation that describes the fall and rise of the water \((w: \text{wave height})\) as the waves go by \((x: \text{distance})\) in time \( t \) can be represented by the one-dimensional wave equation as follows:
\[ \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \]
Show that \( w(x,t) = 5\cos(3x + 3ct) + e^{x+ct} \) satisfies the wave equation given above where \( c \) is the velocity with which the waves are propagated.

10. a) (15 p) Express \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) as the following function of \( u \) and \( v \) both by using the Chain Rule and by expressing \( w \) directly in terms of \( u \) and \( v \) before differentiating:
\[ w = \ln(x^2 + y^2 + z^2) \quad ; \quad x = u e^v \sin u \quad ; \quad y = u e^v \cos u \quad ; \quad z = u e^v \]

b) (5 p) Evaluate \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) at the given point \((u, v) = (-2, 0)\).

Taylor’s Formula:
\[ f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{n}(a)}{n!}(x - a)^n + R_n(x) \]
Apply IBP (Integration by parts) as follows:

\[ \int u \, dv = uv - \int v \, du \]
\[ u = \cos 2x, \quad du = -2 \sin 2x \, dx; \quad dv = e^x \, dx, \quad v = e^x \]

\[ l = \int \cos 2x \, e^x \, dx = (\cos 2x)(e^x) - \int e^x(-2 \sin 2x) \, dx = e^x(\cos 2x) + 2 \int e^x \sin 2x \, dx \]

Apply again IBP for the inside integral called \( A = \int e^x \sin 2x \, dx \) as follows:
\[ u = \sin 2x, \quad du = 2 \cos 2x \, dx; \quad dv = e^x \, dx, \quad v = e^x \]

\[ A = \int e^x \sin 2x \, dx = (\sin 2x)(e^x) - 2 \int e^x(\cos 2x) \, dx = e^x(\sin 2x) - 2 \int e^x \cos 2x \, dx \]
\[ \int \cos 2x \, e^x \, dx = e^x(\cos 2x) + 2 \left( e^x \sin 2x - 2 \int e^x \cos 2x \, dx \right) \]
\[ l = e^x \cos 2x + 2e^x \sin 2x - 4l + C' \]
\[ 5l = e^x \cos 2x + 2e^x \sin 2x + C' \]
\[ l = \frac{e^x \cos 2x}{5} + \frac{2e^x \sin 2x}{5} + \frac{C'}{5} = \frac{e^x}{5}(\cos 2x + 2\sin 2x) + C \]

2.

\[ \int \sin^5 x \cos^5 x \, dx = \int \sin^5 x \cos^4 x \cos x \, dx \]

Apply u-substitution as follows:
\[ u = \sin x, \quad du = \cos x \, dx \]
\[ \int \sin^5 x (1 - \sin^2 x)^2 \cos x \, dx = \int u^5(1 - u^2)^2 \, du = \int u^5(1 - 2u^2 + u^4) \, du \]
\[ = \int u^5 du - 2 \int u^7 du + \int u^9 du = \frac{u^6}{6} - 2 \frac{u^8}{8} + \frac{u^{10}}{10} + C \]
\[ = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{4} + \frac{\sin^{10} x}{10} + C \]

3.

Given: \( x = \sqrt{5 - \sqrt{t}} \); \( y(t - 1) = \sqrt{t} \); \( t = 4 \)

Explicit and implicit differentiation of these equations respectively gives:
\[ \frac{dx}{dt} = \frac{1}{2} \frac{5 - \sqrt{t}}{5 - \sqrt{t}} \cdot \left( -\frac{1}{2} t^{-1/2} \right) = -\frac{1}{4\sqrt{t}} \frac{1}{5 - \sqrt{t}} \]

\[ (t - 1) \frac{dy}{dt} + y = \frac{1}{2} t^{-1/2} \rightarrow (t - 1) \frac{dy}{dt} = \frac{1}{2\sqrt{t}} - y \rightarrow \frac{dy}{dt} = \frac{1 - 2y\sqrt{t}}{2\sqrt{t} - 2\sqrt{t}} = \frac{1}{2\sqrt{t}} - \frac{y}{t - 1} = \frac{1 - 2y\sqrt{t}}{2\sqrt{t}(t - 1)} \]

\[ \frac{dy}{dx} = -\frac{\frac{1 - 2y\sqrt{t}}{2\sqrt{t} - 2\sqrt{t}}}{\frac{1}{4\sqrt{t}} \frac{5 - \sqrt{t}}{5 - \sqrt{t}}} = \frac{1 - 2y\sqrt{t}}{2\sqrt{t} - 2\sqrt{t}} \frac{4\sqrt{t}}{(5 - \sqrt{t})} \left( \frac{5 - \sqrt{t}}{1 - t} \right) = \frac{2(1 - 2y\sqrt{t})(5 - \sqrt{t})}{1 - t} ; \]

\[ t = 4 \rightarrow x = \sqrt{5 - 4} = x = \sqrt{3}; \]

\[ t = 4 \rightarrow y(4 - 1) = \sqrt{4} - y = \frac{2}{3} \]
\[
\frac{dy}{dx}_{t=4} = \frac{2(1 - 2y\sqrt{t})\sqrt{5 - \sqrt{t}}}{1 - t} = \frac{2 \left( 1 - 2 \frac{2}{3}\sqrt{4} \right) \sqrt{5 - \sqrt{4}}}{1 - 4} = \frac{10\sqrt{3}}{9}
\]

4. The length of the curve given by the polar coordinate equation as follows:
\[
r = 2\sin \theta + 2\cos \theta; 0 \leq \theta \leq \frac{\pi}{2} \rightarrow \frac{dr}{d\theta} = 2\cos \theta - 2\sin \theta
\]
\[
L = \int_0^{\pi/2} \sqrt{(2\sin \theta + 2\cos \theta)^2 + (2\cos \theta - 2\sin \theta)^2} \, d\theta = \int_0^{\pi/2} \sqrt{8(\sin^2 \theta + \cos^2 \theta)} \, d\theta
\]
\[
= \int_0^{\pi/2} 2\sqrt{2} \, d\theta = \left[ 2\sqrt{2} \frac{\theta}{2} \right]_0^{\pi/2} = \pi \sqrt{2}
\]
The area of the curve:
\[
r = 2\sin \theta + 2\cos \theta; 0 \leq \theta \leq \frac{\pi}{2} \rightarrow \frac{dr}{d\theta} = 2\cos \theta - 2\sin \theta
\]
\[
A = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{\pi/2} (2\sin \theta + 2\cos \theta)^2 \, d\theta = \frac{1}{2} \int_0^{\pi/2} (4 \sin^2 \theta + 8 \sin \theta \cos \theta + 4 \cos^2 \theta) \, d\theta
\]
\[
= \frac{1}{2} \int_0^{\pi/2} (4 + 8 \sin \theta \cos \theta) \, d\theta
\]
\[
= 2 \int_0^{\pi/2} \sin \theta \, d\theta + 4 \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \left[ 2\theta + 4 \left( \frac{\sin \theta \cos \theta}{2} \right) \right]_0^{\pi/2}
\]
(substitution \( u = \sin \theta \))
\[
= \pi + 2
\]

5. \( f(x) = \frac{1}{x + 4} \)
is positive, continuous, and decreasing for \( x \geq 1 \).
\[
\int_1^\infty \frac{1}{x + 4} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x + 4} \, dx = \lim_{b \to \infty} [\ln|x + 4|]_1^b = \lim_{b \to \infty} (\ln|b + 4| - \ln 5) = \infty
\]
\[
\int_1^\infty \frac{1}{x + 4} \, dx \text{ diverges} \rightarrow \sum_{n=1}^\infty \frac{1}{n + 4} \text{ diverges}
\]

In the evaluation of the integral u-substitution \((u = x + 4)\) was used.

6. The series given converges by the Ratio Test as follows:
\[
\sum_{n=1}^\infty n^2 e^{-n}
\]
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n + 1)^2 e^{-(n+1)}}{n^2 e^{-n}} = \lim_{n \to \infty} \frac{(n + 1)^2}{(n)^2} \left( \frac{e}{(1 + \frac{1}{n})} \right) \left( \frac{1}{e} \right) = \frac{1}{e} < 1
\]

7. Taylor’s Formula:
\[
f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^n(a)}{n!}(x - a)^n + R_n(x)
\]
\[
R_n(x) = \frac{f^{n+1}(c)}{(n + 1)!}(x - a)^{n+1} \text{ for some } c \text{ between } a \text{ and } x
\]

Apply Taylor’s formula given above to the function up to \( 3^{rd} \) order derivative as follows:
Zero, first, second and third order polynomials generated by Taylor’s formula were given respectively as follows:

\[ f(x) = \ln(1 + x) \; ; \; a = 0 \rightarrow f(a) = f(0) = \ln 1 = 0 ; \; f'(x) = \frac{1}{1 + x} \rightarrow f'(a) = \frac{1}{1} = 1 \]

\[ f'(0) = 1 \; ; \; f''(x) = -(1 + x)^{-2} \rightarrow f''(a) = f''(0) = -(1)^{-2} = -1 ; \; f'''(x) = 2(1 + x)^{-3} \rightarrow f'''(a) = f'''(0) = 2(1)^{-3} = 2 \]

Thus

Zeros, first, second and third order polynomials generated by \( f(x) = \ln(1 + x) \) according to Taylor’s formula were given respectively as follows:

\[ P_0(x) = f(a) = 0 \; ; \; P_1(x) = f(a) + f'(a)(x - a) = 0 + (1)(x - 0) = x \]

\[ P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 = 0 + (1)(x - 0) + \left( \frac{-1}{2} \right) (x - 0)^2 = x - \frac{1}{2} x^2 \]

\[ P_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 = P_2(x) + \frac{2}{6} (x - 0)^3 \]

\[ = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 \]

8. Maclaurin series generated by \( f(x) = xe^x \) is:

\[ \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} (x)^n = f(0) + \frac{f'(0)}{2!} x^2 + \ldots + \frac{f^n(0)}{n!} x^n + \ldots \]

We need to find \( f(0), f'(0), f''(0), \ldots \) according to the formula given above as follows:

\[ f(x) = xe^x ; \; f'(x) = e^x + xe^x ; \; f''(x) = e^x + e^x + xe^x = 2e^x + xe^x ; \; f'''(x) = 3e^x + xe^x \]

\[ f(0) = 0e^0 = 0 ; \; f'(0) = e^0 + 0e^0 = 1 ; \; f''(0) = 2e^0 + 0e^0 = 2 ; \; f'''(0) = 3e^0 + 0e^0 = 3 ; \ldots \]

\[ \sum_{n=0}^{\infty} \frac{n}{n!} (x)^n = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} x^n = x + x^2 + \frac{1}{2} x^3 + \ldots + \frac{n}{n!} x^n + \ldots \]

9. To show the following equation given to be a solution of the partial differential equation, we need to take second order partial derivative according to \( t \) and second order partial derivative according to \( x \) as follows:

\[ w(x, t) = 5\cos(3x + 3ct) + e^{3x + 3ct} \]

\[ \frac{\partial w}{\partial t} = -15c \sin(3x + 3ct) + ce^{3x + 3ct} \]

\[ \frac{\partial w}{\partial x} = -15 \sin(3x + 3ct) + e^{3x + 3ct} \]

\[ \frac{\partial^2 w}{\partial x^2} = -45\cos(3x + 3ct) + e^{3x + 3ct} \]

\[ \frac{\partial^2 w}{\partial t^2} = -45c^2 \cos(3x + 3ct) + c^2 e^{3x + 3ct} \]

Then

\[ \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \]

\[ -45c^2 \cos(3x + 3ct) + c^2 e^{3x + 3ct} = c^2(-45\cos(3x + 3ct) + e^{3x + 3ct}) \]

Thus

\[ \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \]
10. a) By using Chain Rule:
\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}
\]
\[
= \left(\frac{2x}{x^2 + y^2 + z^2}\right)(e^v \sin u + ue^v \cos u) + \left(\frac{2y}{x^2 + y^2 + z^2}\right)(e^v \cos u - ue^v \sin u)
\]
\[
+ \left(\frac{2z}{x^2 + y^2 + z^2}\right)(e^v)
\]
\[
= \left(\frac{2ue^v \sin u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(e^v \sin u + ue^v \cos u)
\]
\[
+ \left(\frac{2ue^v \cos u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(e^v \cos u - ue^v \sin u)
\]
\[
+ \left(\frac{2ue^v}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(e^v) = \frac{2}{u}
\]

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}
\]
\[
= \left(\frac{2x}{x^2 + y^2 + z^2}\right)(ue^v \sin u) + \left(\frac{2y}{x^2 + y^2 + z^2}\right)(ue^v \cos u)
\]
\[
+ \left(\frac{2z}{x^2 + y^2 + z^2}\right)(ue^v)
\]
\[
= \left(\frac{2ue^v \sin u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(ue^v \sin u)
\]
\[
+ \left(\frac{2ue^v \cos u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(ue^v \cos u)
\]
\[
+ \left(\frac{2ue^v}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(ue^v) = 2
\]

By expressing \( w \) directly in terms of \( u \) and \( v \) before differentiating:
\[
w = \ln(x^2 + y^2 + z^2) = \ln(u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}) = \ln(2u^2 e^{2v})
\]
\[
= \ln 2 + 2 \ln u + 2v
\]
\[
x = ue^v \sin u ; y = ue^v \cos u ; z = ue^v
\]

Differentiation of the last expression gives the same result as follows:
\[
\frac{\partial w}{\partial u} = \frac{2}{u} ; \quad \frac{\partial w}{\partial v} = \frac{2}{u}
\]

b)\[
\frac{\partial w}{\partial u} \bigg|_{(u,v)} = \frac{\partial w}{\partial u} \bigg|_{(-2,0)} = \frac{2}{-2} = -1
\]
\[
\frac{\partial w}{\partial v} \bigg|_{(u,v)} = \frac{\partial w}{\partial v} \bigg|_{(-2,0)} = 2
\]