1. (25 p) The Beattie-Bridgeman equation of state
\[ P = \frac{RT}{V} + \frac{a}{V^2} + \frac{b}{V^3} + \frac{c}{V^4} \]
is a three-parameter extension of the ideal gas law. Using \( a = -1.06, b = 0.057, \text{and } c = -0.0001 \) find the volume of 1 mole of a gas at \( P = 25 \text{ atm}, \text{and } T = 293 \text{ K} \). The constant \( R = 0.082 \text{ L.atm}/(\text{mol.K}) \) is the temperature in °C. Use Bisection method with \( V_l = 0.9 \frac{L}{mol}, \)
\[ V_u = 1 \frac{l}{mol} \text{ and } \epsilon_s = 3%. \]

2. (25 p) Three blocks are connected by a weightless cord and rest on an inclined plane. Employing a procedure (free-body diagrams for each block) by using Newton’s second law gives a set of the following three simultaneous linear equations:
\[ 100a + T = 519.72 \]
\[ 50a - T + R = 216.55 \]
\[ 25a - R = 108.27 \]
Solve for acceleration \( a \), and tensions \( T \) and \( R \) in the two ropes. Use Gauss-Seidel method with the initial values of \( a^0 = 4 \frac{m}{s^2}, T^0 = 35 \text{ N}, R^0 = 10 \text{ N } \text{and } \epsilon_s = 10\% \).

3. (25 p) Enzymatic reactions are used extensively to characterize biologically mediated reactions. The following is an example of a model (saturation-growth-rate type) that is used to fit such reactions:
\[ v_0 = \frac{k_m[S]^3}{K + [S]^3} \]
where \( v_0 = \text{initial rate of reaction } (M/s), [S] = \text{the substrate concentration } (M) \).
\( k_m \) and \( K \) are parameters to be found by fitting the following data to this model:

<table>
<thead>
<tr>
<th>( [S], \text{M} )</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_0, M/s )</td>
<td>6.078x10^{-11}</td>
<td>6.063x10^{-8}</td>
<td>1.737x10^{-5}</td>
<td>2.430x10^{-5}</td>
<td>2.431x10^{-5}</td>
</tr>
</tbody>
</table>

(a) Use a transformation to linearize the model (b) Evaluate the parameters \( k_m \) and \( K \) by using linear least-squares regression and estimate the initial rate of reaction \( v_0 \) as \( M/s \) for the substrate concentration of \( [S] = 5 \text{ M} \).

4. (25 p) Evaluate the following integral:
\[ \int_{40}^{93} \frac{97000v}{(5v^2 + 570000)} \, dv \]

a) Analytically, b) By using composite Simpson’s 1/3 Rule with \( n = 4 \text{ segments} \).

c) Calculate absolute relative true error as percentage (\( \epsilon_t, \% \)).

5. (25 p) a) Determine the value of \((-2 - j)^{1/2}\), expressing the result in polar and rectangular forms.\( b)\) Determine the moduli and arguments of the complex square roots \((-6 - j5)^{1/2}\)

Recall: \( a_1 = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \); \( a_0 = \bar{y} - a_1 \bar{x} \)

\[ I \equiv (b - a) \frac{f(x_0) + 4 \sum_{i=1,3,5}^{n-1} f(x_i) + 2 \sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n} \]
1. By substituting the values given in the Beattie-Bridgeman equation of state, the solution can be formulated as follows:

\[
25 = \frac{0.082(293)}{V} + \frac{(-1.06)}{V^2} + \frac{0.057}{V^3} + \frac{(-0.0001)}{V^4}
\]

\[
\rightarrow f(V) = \frac{24.026}{V} - \frac{1.06}{V^2} + \frac{0.057}{V^3} - \frac{1\times 10^{-4}}{V^4} - 25 = 0
\]

Applying Bisection method with the initial guesses of \(V_l = 0.9 \text{ L/mol} \); \(V_u = 1 \text{ L/mol}\) gives:

Check: \(f(V_l)f(V_u) = f(0.9)f(1) = (0.46495)(-1.9771) < 0\)

**Iteration 1: Bracket [0.9; 1]**

\[
V_r = \frac{V_l + V_u}{2} = \frac{0.9 + 1}{2} = 0.95
\]

\[
f(V_l)f(V_r) = f(0.9)f(0.95) = (0.46495)(-0.8176) < 0
\]

The root should be between \(V_l = 0.9 \); \(V_r = V_u = 0.95\).

**Iteration 2: Bracket [0.9; 0.95]**

\[
V_r = \frac{V_l + V_u}{2} = \frac{0.9 + 0.95}{2} = 0.925
\]

\[
\epsilon_a = \left| \frac{0.925 - 0.95}{0.925} \right| \times 100 = 2.7\% < \epsilon_s (3\%) \text{ STOP}
\]

Root \(V = 0.925\) with an absolute relative approximate error of 2.7%

Note: If continued, Iteration 3 bracket will be: [0.9; 0.925], because \(f(V_l)f(V_r) = f(0.9)f(0.925) = (0.46495)(-0.1929) < 0, V_r = 0.9125, \epsilon_a = 1.37\%

2. The system of linear algebraic equations to be solved by using Gauss-Seidel method:

\[
\begin{align*}
100a + T &= 519.72 \\
50a - T + R &= 216.55 \\
25a - R &= 108.27
\end{align*}
\]

**Iteration 0 (Initial values):**

\(a^0 = 4 \text{ m/s}^2; T^0 = 35 \text{ N}; R^0 = 10 \text{ N}\)

**Iteration 1:**

\[
a^1 = \frac{519.72 - T^0}{100} = \frac{519.72 - 35}{100} = 4.8472
\]

\[
T^1 = \frac{216.55 - 50a^1 - R^0}{-1} = \frac{216.55 - 50(4.8472) - 10}{-1} = 35.81
\]

\[
R^1 = \frac{108.27 - 25a^1}{-1} = \frac{108.27 - 25(4.8472)}{-1} = 12.91
\]

The error estimates can be computed as follows:

\[
\epsilon_{a,1}(\%) = \left| \frac{4.8472 - 4}{4.8472} \right| \times 100 = 17.5\% > \epsilon_s (\text{given 10\%})
\]
**Iteration 2:**

\[
\alpha^2 = \frac{519.72 - T^1}{100} = \frac{519.72 - 35.81}{100} = 4.8391
\]

\[
T^2 = \frac{216.55 - 50\alpha^2 - R^1}{-1} = \frac{216.55 - 50(4.8391) - 12.91}{-1} = 38.315
\]

\[
R^2 = \frac{108.27 - 25\alpha^2}{-1} = \frac{108.27 - 25(4.8391)}{-1} = 12.7075
\]

The error estimates can be computed as follows:

\[
\varepsilon_{a,1} = \frac{4.8391 - 4.8472}{4.8391} \times 100 = 0.17\% < 10\%
\]

\[
\varepsilon_{a,2} = \frac{38.315 - 35.81}{38.315} \times 100 = 6.54\% (\text{maximum}) < 10\% \text{ STOP}
\]

\[
\varepsilon_{a,3} = \frac{12.7075 - 12.91}{12.7075} \times 100 = 1.6\% < 10\%
\]

The solution vector \((\alpha, T, R)\) is as follows:

\[
\begin{bmatrix}
\alpha \\
T \\
R
\end{bmatrix} =
\begin{bmatrix}
4.8391 \\
38.315 \\
12.7075
\end{bmatrix}
\]

3.a) Employing saturation – growth-rate model requires transformation (linearization) of the model to use least-square method as follows:

\[
v_0 = \frac{k_m[S]^3}{K + [S]^3}
\]

\[
\frac{1}{v_0} = \frac{1}{k_m[S]^3} + \frac{1}{k_m} \rightarrow z = a_0 + a_1 w \rightarrow z = \frac{1}{v_0} ; a_0 = \frac{1}{k_m} ; a_1 = \frac{K}{k_m} ; w = \frac{1}{[S]^3}
\]

Apply Least-Square method for the transformed form of saturation-growth-rate model to find regression coefficients as follows:

\[
a_i = \frac{n \sum_{i=1}^{n} w_{i} z_{i} - \sum_{i=1}^{n} w_{i} \sum_{i=1}^{n} z_{i}}{n \sum_{i=1}^{n} w_{i}^2 - (\sum_{i=1}^{n} w_{i})^2} ; a_0 = \bar{z} - a_1 \bar{w}
\]

**Table.** Summations of data to calculate coefficients of the linearized model:

<table>
<thead>
<tr>
<th>(i)</th>
<th>([S]^3)_i</th>
<th>((v_0)_i)</th>
<th>(z_i = \left[ \frac{1}{v_0}_i \right] )</th>
<th>(w_i = \left( \frac{1}{[S]^3}\right)_i )</th>
<th>(w_i z_i )</th>
<th>(w_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1x10^-6</td>
<td>6.078x10^-11</td>
<td>1.6453x10^16</td>
<td>1x10^6</td>
<td>1.6453x10^16</td>
<td>1x10^12</td>
</tr>
<tr>
<td>2</td>
<td>1x10^-3</td>
<td>6.063x10^-8</td>
<td>1.6494x10^10</td>
<td>1x10^3</td>
<td>1.6494x10^10</td>
<td>1x10^6</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1.737x10^-5</td>
<td>5.7571x10^4</td>
<td>1</td>
<td>5.7571x10^4</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1x10^-3</td>
<td>2.430x10^-5</td>
<td>4.1152x10^4</td>
<td>1x10^3</td>
<td>4.1152x10^4</td>
<td>1x10^6</td>
</tr>
<tr>
<td>5</td>
<td>1x10^-6</td>
<td>2.431x10^-5</td>
<td>4.1135x10^4</td>
<td>1x10^-6</td>
<td>4.1135x10^-2</td>
<td>1x10^-12</td>
</tr>
<tr>
<td>(\Sigma)</td>
<td>1.647x10^10</td>
<td>1.001x10^6</td>
<td>1.6453x10^16</td>
<td></td>
<td>1.000x10^12</td>
<td></td>
</tr>
</tbody>
</table>

By substituting the corresponding values at the bottom of the table to the least square equations given above, we have:
\[ a_1 = \frac{5(1.6453 \times 10^{16}) - (1.001 \times 10^6)(1.647 \times 10^{10})}{5(1.000 \times 10^{12}) - (1.001 \times 10^6)^2} = 1.645286 \times 10^4 \]

\[ a_0 = \frac{(1.647 \times 10^{10})}{5} - (1.645286 \times 10^{4}) \frac{1.001 \times 10^6}{5} \times 1.37428 \times 10^5 = \frac{1}{k_m} \rightarrow k_m = \frac{1}{1.37428 \times 10^5} = 7.2765 \times 10^{-6}; K = \frac{a_1}{a_0} = 1.645286 \times 10^4 \frac{1.001 \times 10^6}{1.37428 \times 10^5} = 0.1197 \]

b) Saturation-growth-rate model to estimate \( v_0 \) value for a given \([S]\) value of 5 M:

\[ v_0 = \frac{k_m [S]^3}{K + [S]^3} = \frac{7.2765 \times 10^{-6}[S]^3}{0.1197 + [S]^3} = \frac{7.2765 \times 10^{-6}(5)^3}{0.1197 + (5)^3} = 7.2695 \times 10^{-6} \text{ M/s} \]

4.a) Analytically:
Apply u-substitution as follows:

\[ u = (5v^2 + 570000), du = 10v dv \]

\[
\int_{40}^{93} \frac{97000v}{(5v^2 + 570000)} dv = \frac{97000}{10} \int_{40}^{93} \frac{10v}{(5v^2 + 570000)} dv = 9700 \int_{[5(40)^2 + 570000]}^{[5(93)^2 + 570000]} \frac{du}{u} = 9700 [\ln u]_{613245}^{578000} = 9700(\ln 613245 - \ln 578000) = 574.15 \text{ (True value)}
\]

b) Applying Composite Simpson’s 1/3 Rule:

\[
l \approx (b - a) \frac{f(x_0) + 4 \sum_{i=1,3,5} f(x_i) + 2 \sum_{j=2,4,6} f(x_j) + f(x_n)}{3n}
\]

\[ a = x_0 = 40, b = x_n = x_4 = 93, n = 4 \text{ (given)}, h = \frac{b - a}{n} = \frac{93 - 40}{4} = 13.25, x_1 = 53.25, x_2 = 66.5, x_3 = 79.75, x_4 = 93
\]

\[ I \approx (93 - 40) \frac{f(40) + 4[f(53.25) + f(79.75)] + 2f(66.5) + f(93)}{12} = 574.15
\]

c) \((\varepsilon_t, \%)\):

\[ \varepsilon_t (\%) = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 = \left| \frac{574.15 - 574.15}{574.15} \right| \times 100 = 0 \%
\]

5.a)

\[-2 - j = \sqrt{(-2)^2 + (-1)^2} \tan^{-1} \frac{-1}{-2} = \sqrt{5} \angle -153.43^\circ \text{ (3rd Quadrant angle)} \]

Applying De Moivre’s theorem:

\[ (-2 - j)^5 = [\sqrt{5} \angle -153.43^\circ]^5 = (\sqrt{5})^5 \angle 5(-153.43^\circ) = 55.90 \angle -767.15^\circ \]
By adding 360° (counterclockwise) two times, equivalent angle will be \(-47.15°\) (4th Quadrant angle). The final result will be given in polar form as follows:

\[
(-2 - j)^5 = 55.90 \angle -47.15° \text{ (Polar form)}
\]

For rectangular form:

\[
r \angle \theta = r(\cos \theta + j \sin \theta)
\]

\[
55.90 \angle -47.15° = 55.90[\cos(-47.15°) + j \sin(-47.15°)] = 55.90[0.680081 + j(-0.7331)]
\]

\[
= 38 - 41j
\]

b) Determine the moduli and arguments of the complex roots:

\[
(-6 - j5)^{1/2} = (x)^{1/2} = \sqrt[2]{x} \text{ (complex square roots)}
\]

The roots are symmetrically displaced from one another \(\frac{360°}{n} = \frac{360°}{2} = 180°\) apart round an Argand diagram as follows:

\[
(-6 - j5) = \sqrt{(-6)^2 + (-5)^2} \tan^{-1} \frac{-5}{-6} = \sqrt{61} \angle -140.19° \text{ (3rd Quadrant angle)}
\]

Applying De Moivre’s theorem:

\[
(-6 - j5)^{1/2} = \left[\sqrt{61} \angle -140.19°\right]^{1/2} = (61)^{1/4} \angle \left(\frac{1}{2}\right)(-140.19°) = 2.795 \angle -70.095° \text{ (1st root in 4th Quadrant)},
\]

\[
2.795 \angle (-70.095° + 180°) = 2.795 \angle 109.905° \text{ (2nd root in 2nd Quadrant)}
\]