1. (35 p) The Ergun equation, shown below, is used to describe the flow of a fluid through a packed bed. $\Delta P$ is the pressure drop, $\rho$ is the density of a fluid, $G_0$ is the mass velocity (mass flow rate divided by cross-sectional area), $D_p$ is the diameter of the particles within the bed, $N_{Re}$ is the dimensionless Reynolds number where $\mu$ is the fluid viscosity, $L$ is the length of the bed, and $\theta$ is the void fraction of the bed:

$$\frac{\Delta P \rho D_p}{G_0^2 L} \frac{\theta^3}{1 - \theta} = 150 \frac{1 - \theta}{N_{Re}} + 1.75$$

Given the parameter values listed below, find the void fraction $\theta$ of the bed by using Bisection method with the $\theta_l = 0.2$; $\theta_u = 0.4$ and $\varepsilon_s = 5\%$.

$$\frac{\Delta P \rho D_p}{G_0^2 L} = 20; \quad N_{Re} = \frac{D_p G_0}{\mu} = 1000$$

2. (35 p) An electronics company produces transistors, resistors, and computer chips. Each transistor requires 4 units of Copper, 1 unit of Zinc, and 2 units of Glass. Each resistor and computer chip requires the same materials as units given in the table below:

<table>
<thead>
<tr>
<th>Component</th>
<th>Copper</th>
<th>Zinc</th>
<th>Glass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transistors</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Resistors</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Computer chips</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Supplies of these materials vary from week to week, so the company needs to determine a different production run each week. For example, one week the total amounts of materials available are 960 units of Copper, 510 units of Zinc, and 610 units of Glass.

a) Set up the system of linear algebraic equations modeling the production run,

b) Use Gauss-Seidel method to solve for the number of transistors ($x_1$), resistors ($x_2$), and computer chips ($x_3$) to be manufactured this week. Use initial guesses of unknowns $x_1^0 = x_2^0 = x_3^0 = 100$ units with the stopping criterion $\varepsilon_s = 5\%$.

3. (30 p) A material is tested for cyclic fatigue failure whereby a stress, in MPa, is applied to the material and the number of cycles needed to cause failure is measured. The results are in the table below. Fit a power model to this data by using Least-Square method with the appropriate transformation:

<table>
<thead>
<tr>
<th>N, cycles</th>
<th>Stress, MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1100</td>
</tr>
<tr>
<td>10</td>
<td>1000</td>
</tr>
<tr>
<td>$10^2$</td>
<td>925</td>
</tr>
<tr>
<td>$10^3$</td>
<td>800</td>
</tr>
<tr>
<td>$10^4$</td>
<td>625</td>
</tr>
<tr>
<td>$10^5$</td>
<td>550</td>
</tr>
<tr>
<td>$10^6$</td>
<td>420</td>
</tr>
</tbody>
</table>

Recall:

$$Power \ model \rightarrow Stress = a(N)^\beta; \quad a_1 = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}; \quad a_0 = \bar{y} - a_1 \bar{x}$$
1. By substituting the values given in the Ergun Equation, the solution can be formulated as follows:

\[
\frac{20}{\theta} \frac{\theta^3}{1-\theta} = 150 \frac{1-\theta}{1000} + 1.75 \rightarrow f(\theta) = 0.15(1-\theta) + 1.75 - 20 \frac{\theta^3}{1-\theta} = 0
\]

Applying Bisection method with the initial guesses of \( \theta_l = 0.2; \theta_u = 0.4 \) gives:

**Iteration 1: Bracket [0.2; 0.4]**

\[
\theta_r = \frac{\theta_l + \theta_u}{2} = \frac{0.2 + 0.4}{2} = 0.3
\]

\[f(\theta_l)f(\theta_r) = f(0.2)f(0.3) = (1.67)(1.083) = 1.8086 > 0\]

\[f(\theta_r)f(\theta_u) = f(0.3)f(0.4) = (1.083)(-0.293) = -0.317 < 0\]

The root cannot be between \( \theta_l = 0.2; \theta_r = 0.3 \). Consequently, the root is in the second interval as shown above. The lower guess and upper guess are redefined as \( \theta_r = \theta_l = 0.3; \theta_u = 0.4 \)

**Iteration 2: Bracket [0.3; 0.4]**

\[
\theta_r = \frac{\theta_l + \theta_u}{2} = \frac{0.3 + 0.4}{2} = 0.35
\]

\[f(\theta_l)f(\theta_r) = f(0.3)f(0.35) = (1.083)(0.5283) = +0.5722 > 0\]

The root cannot be between \( \theta_l = 0.3; \theta_r = 0.35 \). Consequently, the root is in the second interval and the lower guess and upper guess are redefined as \( \theta_r = \theta_l = 0.35; \theta_u = 0.4 \)

\[
\varepsilon_a = \left| \frac{0.35 - 0.35}{0.35} \right| \times 100 = 14.3% > \varepsilon_s (5%) \]

**Iteration 3: Bracket [0.35; 0.4]**

\[
\theta_r = \frac{\theta_l + \theta_u}{2} = \frac{0.35 + 0.4}{2} = 0.375
\]

\[f(\theta_l)f(\theta_r) = f(0.35)f(0.375) = (0.5283)(0.1563) = +0.0826 > 0\]

The root cannot be between \( \theta_l = 0.35; \theta_r = 0.375 \). Consequently, the root is in the second interval and the lower guess and upper guess are redefined as \( \theta_r = \theta_l = 0.375; \theta_u = 0.4 \)

\[
\varepsilon_a = \left| \frac{0.375 - 0.35}{0.375} \right| \times 100 = 6.7% > 5% \]

**Iteration 4: Bracket [0.375; 0.4]**

\[
\theta_r = \frac{\theta_l + \theta_u}{2} = \frac{0.375 + 0.4}{2} = 0.3875
\]

\[f(\theta_l)f(\theta_r) = f(0.375)f(0.3875) = (0.1563)(-0.05806) = -9.07 \times 10^{-3} < 0\]

The root \( \theta = 0.3875 \) with an absolute relative approximate error of 3.23%

Note: If continued, Iteration 5 bracket will be: \([0.375; 0.3875]\), because \( f(\theta_l)f(\theta_r) = f(0.375)f(0.3875) = (0.1563)(-0.05806) = -9.07 \times 10^{-3} < 0 \)
2. a) The system of linear algebraic equations modeling the production run can be set up by using table data given as follows:
   \[\begin{align*}
   4x_1 + 3x_2 + 2x_3 &= 960 \\
   x_1 + 3x_2 + x_3 &= 510 \\
   2x_1 + x_2 + 3x_3 &= 610
   \end{align*}\]

b) Use Gauss-Seidel to determine unknowns as follows:

**Iteration 0:**
\[x_1^0 = x_2^0 = x_3^0 = 100 \text{ units}\]

**Iteration 1:**
\[\begin{align*}
  x_1 &= \frac{960 - 3x_2 - 2x_3}{4} = \frac{960 - 3(100) - 2(100)}{4} = 115 \\
  x_2 &= \frac{510 - x_1 - x_3}{3} = \frac{510 - 115 - 100}{3} = 98.333 \\
  x_3 &= \frac{610 - 2x_1 - x_2}{3} = \frac{610 - 2(115) - 98.333}{3} = 93.889
\end{align*}\]

The error estimates can be computed as follows:
\[\begin{align*}
  \varepsilon_{a,1} &= \left| \frac{115 - 100}{115} \right| \times 100 = 13\% > 5\% \\
  \varepsilon_{a,2} &= \left| \frac{98.333 - 100}{98.333} \right| \times 100 = 1.7\% \\
  \varepsilon_{a,3} &= \left| \frac{93.889 - 100}{93.889} \right| \times 100 = 6.5\% > 5\%
\end{align*}\]

**Iteration 2:**
\[\begin{align*}
  x_1 &= \frac{960 - 3x_2 - 2x_3}{4} = \frac{960 - 3(98.333) - 2(93.889)}{4} = 119.31 \\
  x_2 &= \frac{510 - x_1 - x_3}{3} = \frac{510 - 119.31 - 93.889}{3} = 98.934 \\
  x_3 &= \frac{610 - 2x_1 - x_2}{3} = \frac{610 - 2(119.31) - 98.934}{3} = 90.815
\end{align*}\]

The error estimates can be computed as follows:
\[\begin{align*}
  \varepsilon_{a,1} &= \left| \frac{119.31 - 115}{119.31} \right| \times 100 = 3.61\% \text{ (maximum) < 5\%} \\
  \varepsilon_{a,2} &= \left| \frac{98.934 - 98.333}{98.934} \right| \times 100 = 0.61\% < 5\% \\
  \varepsilon_{a,3} &= \left| \frac{90.815 - 93.889}{90.815} \right| \times 100 = 3.39\% < 5\%
\end{align*}\]

The solution vector (the number of transistors, resistors, and computer chips to be manufactured this week respectively) is as follows:
\[\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 119.31 \\ 98.934 \\ 90.815 \end{bmatrix}\]
3. Taking **power model** relationship between the dependent variable **Stress** and the independent variable **N** to regress the data:

\[
\text{Stress} = aN^\beta \rightarrow \log_{10}(\text{Stress}) = \log_{10}a + \beta \log_{10}N \rightarrow Y_i = \log_{10}(\text{Stress})_i ; a_0 = \log_{10}a; a_1 = \beta; X_i = \log_{10}N \rightarrow \text{Least - square method to be applied } Y_i = a_0 + a_1X_i \rightarrow \alpha = 10^{a_0}; \beta = a_1
\]

Apply Least-Square method to find regression coefficients as follows:

\[
a_1 = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}
\]

\[
a_0 = \bar{y} - a_1 \bar{x}
\]

**Table. Summations of data to calculate coefficients of the power model**

<table>
<thead>
<tr>
<th>i</th>
<th>(N_i)</th>
<th>(\text{(Stress)}_i)</th>
<th>(X_i)</th>
<th>(Y_i)</th>
<th>(X_iY_i)</th>
<th>(X_i^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1100</td>
<td>0</td>
<td>3.0414</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1000</td>
<td>1</td>
<td>3.0000</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>(10^2)</td>
<td>925</td>
<td>2</td>
<td>2.9961</td>
<td>5.9322</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>(10^3)</td>
<td>800</td>
<td>3</td>
<td>2.9031</td>
<td>8.7093</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>(10^4)</td>
<td>625</td>
<td>4</td>
<td>2.7959</td>
<td>11.1836</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>(10^5)</td>
<td>550</td>
<td>5</td>
<td>2.7404</td>
<td>13.702</td>
<td>25</td>
</tr>
<tr>
<td>7</td>
<td>(10^6)</td>
<td>420</td>
<td>6</td>
<td>2.6233</td>
<td>15.7398</td>
<td>36</td>
</tr>
<tr>
<td>(\Sigma)</td>
<td></td>
<td></td>
<td></td>
<td>21</td>
<td>20.0702</td>
<td>58.2669</td>
</tr>
</tbody>
</table>

By substituting the corresponding values at the bottom of the table to the least square equations given above, we have:

\[
a_1 = \frac{7(58.2669) - (21)(20.0702)}{7(91) - (21)^2} = -0.06942 = \beta
\]

\[
a_0 = \frac{20.0702}{7} - (-0.06942)\frac{21}{7} = 3.0754 \rightarrow \alpha = 10^{3.0754} = 1189.6
\]

Power model to fit the data:

\[
\text{Stress} = \alpha N^\beta = 1189.6N^{-0.06942}
\]