1. (25 p) Conservation of energy can be used to show that:

\[ 0 = \frac{2k_2 d^{5/2}}{5} + \frac{1}{2} k_1 d^2 - mgd - mgh \]

Solve for \( d \), given the following parameter values by using Newton-Raphson method with the initial value of \( d_0 = 0.12 m \) and \( \varepsilon_s = 5\% \):

\[ k_1 = 50000 \frac{g}{s^2}; \quad k_2 = 40 \frac{g}{s^2 m^{1/2}}; \quad m = 90 \ g, \quad g = 9.81 \frac{m}{s^2}; \quad h = 0.45m \]

2. (25 p) The formula for the Cambridge diet, a popular diet in the 1980’s, was based on the following table. Listed in Table below are three of the ingredients in the diet, together with the amounts of certain nutrients supplied by units (100 grams) of each ingredient as follows:

<table>
<thead>
<tr>
<th>Amounts (g) Supplied per unit (100 g) of Ingredient</th>
<th>Amounts (g) Supplied by Cambridge Diet in One Day</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nutrient</td>
<td>Nonfat milk</td>
</tr>
<tr>
<td>Protein</td>
<td>36</td>
</tr>
<tr>
<td>Carbohydrate</td>
<td>52</td>
</tr>
<tr>
<td>Fat</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Find some combinations of nonfat milk, soy flour, and whey to provide the exact amounts of protein, carbohydrate, and fat supplied by the diet in one day. Use LU Factorization method with partial pivoting.

3. (25 p) R&D department of a company is developing an insulin pump for diabetics. To do this, it is important to understand how insulin is cleared from the body after a meal. The concentration of insulin at any time \( t \) is described by the equation:

\[ C = C_0 e^{\beta t/m} \]

where \( C_0 \) is the initial concentration of insulin, \( t \) is the time in minutes, and \( m \) is the mass of the person in kilograms. Write the equation in a linear form, and use linear least-squares regression to determine the constants \( C_0 \) and \( \beta \) for which the function best fits the following data. Use the model equation to determine for a person whose mass is 65 kg how long it will take the concentration of insulin \( C \) of 10 units in minutes.

<table>
<thead>
<tr>
<th>( t ) (min)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m ) (kg)</td>
<td>75</td>
<td>90</td>
<td>80</td>
<td>60</td>
<td>85</td>
</tr>
<tr>
<td>( C ) (unit)</td>
<td>60.3</td>
<td>46.2</td>
<td>29.2</td>
<td>12.2</td>
<td>15.4</td>
</tr>
</tbody>
</table>

4. (25 p) Evaluate the following integral:

\[ \int_0^1 142^x \, dx \]

a) Analytically, b) By using composite Simpson’s 1/3 Rule with \( n = 4 \) segments, c) Calculate absolute relative true error as percentage (\( \varepsilon_t, \% \))

5. (25 p) a) Determine the value of \((-2 + j7)^4\), expressing the result in polar and rectangular forms. b) Determine the moduli and arguments of the complex roots \((-2 + j7)^{1/4}\)

Recall: \( x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}; \quad [A][x] = \{b\}; \quad [l][U] = [A]; \quad [L][d] = \{b\}; \quad [U][x] = \{d\} \)

\[ a_1 = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2}; \quad a_0 = \bar{y} - a_1 \bar{x} \]

\[ I \equiv (b - a) \frac{f(x_0) + 4 \sum_{i=1,3,5} f(x_i) + 2 \sum_{j=2,4,6} f(x_j) + f(x_n)}{3n} \]
MATH 302 NUMERICAL METHODS SPRING 14-15 MAKEUP EXAM SOLUTIONS

Date: 15. 06. 2015  
Instructor: Prof. Dr. Hüseyin Oğuz

1. The solution can be formulated as:
   \[ f(d) = \frac{2k_2 d^{5/2}}{5} + \frac{1}{2} k_1 d^2 - m gd - mgh = 0 \]
   \[ f(d) = \frac{2(40) d^{5/2}}{5} + \frac{1}{2} (50000)d^2 - (90)(9.81)d - (90)(9.81)(0.45) = 0 \]
   \[ f(d) = 16d^{5/2} + 25000d^2 - 882.9d - 397.305 = 0 \]
   \[ f'(d) = 40d^{3/2} + 50000d - 882.9 \]
   
   \[ d_{i+1} = d_i - \frac{f(d_i)}{f'(d_i)} \]

Applying Newton-Raphson formula given above gives with the initial guess of \( d_0 = 0.12 \) m

**Iteration 1:**
   \[ d_1 = d_0 - \frac{f(d_0)}{f'(d_0)} = 0.12 - \frac{16(0.12)^{5/2} + 25000(0.12)^2 - 882.9(0.12) - 397.305}{40(0.12)^{3/2} + 50000(0.12) - 882.9} = 0.12 - \frac{(-143.17)}{5118.76} = 0.14797 \ m \]
   
   \[ \varepsilon_a = \left| \frac{0.14797 - 0.12}{0.14797} \right| \times 100 = 18.9% > 5\% \]

**Iteration 2:**
   \[ d_2 = d_1 - \frac{f(d_1)}{f'(d_1)} = 0.14797 - \frac{16(0.14797)^{5/2} + 25000(0.14797)^2 - 882.9(0.14797) - 397.305}{40(0.14797)^{3/2} + 50000(0.14797) - 882.9} = 0.14797 - \frac{19.561}{6517.88} = 0.14497 \ m \]
   
   \[ \varepsilon_a = \left| \frac{0.14497 - 0.14797}{0.14497} \right| \times 100 = 2.07% < 5\% \]

*Root \( d = 0.145 \) m with at least 1 significant digit*

2. a) Let \( x_i = unit \ of \ ingredient \ i \). The system of linear algebraic equations in order to provide the desired amounts of protein, carbohydrate and fat from the nonfat milk, soy flour and whey in one day can be set up by using table data given as follows:
   \[ 36x_1 + 51x_2 + 13x_3 = 33 \]
   \[ 52x_1 + 34x_2 + 74x_3 = 45 \]
   \[ 0.1x_1 + 7x_2 + 1.1x_3 = 3 \]

b) The system can be written in matrix form with partial pivoting as follows:
   \[ \begin{bmatrix} 52 & 34 & 74 \\ 36 & 51 & 13 \\ 0.1 & 7 & 1.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 33 \\ 3 \end{bmatrix} \]

The solution of the above three simultaneous linear equations by using LU Factorization method will give the value of \( x_1, x_2, x_3 \)

**Forward Elimination of Unknowns:** Since there are three equations, there will be two steps of forward elimination of unknowns.

Prof. Dr. Hüseyin Oğuz

MATH224 MakeupExam-15-06-15
First step: Divide Row 1 by 52 and then multiply it by 36 \((l_{21} = \frac{36}{52} = 0.6923)\) and subtract the result from Row 2:

\[
\text{Row}2 - \left[\frac{\text{Row}1}{52}\right] \times (36) = [0 \quad 27.461 \quad -38.231]
\]

Divide Row 1 by 52 and then multiply it by 0.1 \((l_{31} = \frac{0.1}{52} = 0.00192)\) and subtract the results from Row 3:

\[
\text{Row}3 - \left[\frac{\text{Row}1}{52}\right] \times (0.1) = [0 \quad 6.9346 \quad 0.9577]
\]

\[
[A] = \begin{bmatrix}
52  & 34  & 74 \\
0   & 27.461 & -38.231 \\
0   & 6.9346 & 0.9577 \\
\end{bmatrix}
\]

Second step: We now divide Row 2 by \((27.461)\) and then multiply by \((6.9346)\) \((l_{32} = \frac{6.9346}{27.461} = 0.2525)\) and subtract the results from Row 3:

\[
\text{Row}3 - \left[\frac{\text{Row}2}{27.461}\right] \times (6.9346) = [0 \quad 0 \quad 10.612]
\]

\[
[U] = \begin{bmatrix}
52  & 34  & 74 \\
0   & 27.461 & -38.231 \\
0   & 0   & 10.612 \\
\end{bmatrix}
\]

\[
[L] = \begin{bmatrix}
l_{21} & 1 & 0 \\
l_{31} & 1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
0.6923 & 1 & 0 \\
0.00192 & 0.2525 & 1 \\
\end{bmatrix}
\]

Second step (Forward substitution):

\[ [L][d] = \{b\} \]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0.6923 & 1 & 0 \\
0.00192 & 0.2525 & 1 \\
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
\end{bmatrix}
= \begin{bmatrix}
45 \\
33 \\
3 \\
\end{bmatrix}
\]

\[ d_1 = 45 \]

\[ d_2 = 33 - (0.6923)(45) = 1.8465 \]

\[ d_3 = 3 - 0.00192(45) - 0.2525(1.8465) = 2.4474 \]

Third step (Back substitution): \([U][x] = [d]\)

\[
\begin{bmatrix}
52 & 34 & 74 \\
0   & 27.461 & -38.231 \\
0   & 0   & 10.612 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
45 \\
1.8465 \\
2.4474 \\
\end{bmatrix}
\]

\[ x_3 = \frac{2.4474}{10.612} = 0.231 \text{ units of whey} \]

\[ x_2 = \frac{1.8465 - (-38.231)(0.231)}{27.461} = 0.389 \text{ units of soy flour} \]

\[ x_1 = \frac{45 - 34(0.389) - 74(0.231)}{52} = 0.282 \text{ units nonfat milk} \]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
0.282 \\
0.389 \\
0.231 \\
\end{bmatrix}
\]
3. Employing exponential model requires transformation (linearization) of the model to use least-square method as follows:

\[ C = C_0 e^{\beta t/m} \rightarrow \ln C = \ln(C_0) + \beta \left(\frac{t}{m}\right) \rightarrow z = a_0 + a_1w \]

Apply Least-Square method for the transformed form of exponential model to find regression coefficients as follows:

\[
a_1 = \frac{n \sum_{i=1}^{n} w_i z_i - \left(\sum_{i=1}^{n} w_i\right) \left(\sum_{i=1}^{n} z_i\right)}{n \sum_{i=1}^{n} w_i^2 - \left(\sum_{i=1}^{n} w_i\right)^2} \\
a_0 = \bar{z} - a_1\bar{x}
\]

**Table.** Summations of data to calculate coefficients of the linearized model:

<table>
<thead>
<tr>
<th>i</th>
<th>(t_i) (min)</th>
<th>(m_i) (kg)</th>
<th>(w_i = \left(\frac{t_i}{m_i}\right))</th>
<th>(C_i)</th>
<th>(z_i)</th>
<th>(w_i z_i)</th>
<th>(w_i^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>75</td>
<td>0.0133</td>
<td>60.3</td>
<td>4.099</td>
<td>0.05452</td>
<td>1.7689x10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>90</td>
<td>0.0222</td>
<td>46.2</td>
<td>3.833</td>
<td>0.08509</td>
<td>4.9284x10^{-4}</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>80</td>
<td>0.0375</td>
<td>29.2</td>
<td>3.374</td>
<td>0.1265</td>
<td>1.40625x10^{-3}</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>60</td>
<td>0.0667</td>
<td>12.2</td>
<td>2.501</td>
<td>0.1668</td>
<td>4.44889x10^{-3}</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>85</td>
<td>0.0588</td>
<td>15.4</td>
<td>2.7344</td>
<td>0.1608</td>
<td>3.45744x10^{-3}</td>
</tr>
<tr>
<td>(\Sigma)</td>
<td></td>
<td></td>
<td>0.1985</td>
<td></td>
<td></td>
<td>16.5414</td>
<td>9.98231x10^{-3}</td>
</tr>
</tbody>
</table>

By substituting the corresponding values at the bottom of the table to the least square equations given above, we have:

\[
a_1 = \frac{5(0.5937) - (0.1985)(16.5414)}{5(9.98231x10^{-3}) - (0.1985)^2} = -29.97 = \beta
\]

\[
a_0 = \left(\frac{16.5414}{5}\right) - (-29.97)\left(\frac{0.1985}{5}\right) = 4.4981 = \ln(C_0) \rightarrow C_0 = e^{a_0} = e^{4.4981} = 89.85
\]

Exponential model to estimate the time in minutes for an insulin concentration of 10 units and a mass of person of 65 kg:

\[ C = 89.85e^{-29.97t/m} \rightarrow 10 = e^{-29.97t/65} \rightarrow t = 65 \frac{29.97}{\ln \left(\frac{89.85}{10}\right)} = 4.76 \text{ min} \]

4. a) Analytically:

Apply u-substitution for the second integral as follows:

\[ u = 2x, \; du = 2 \; dx \]

\[
\int_0^1 14^{2x} \; dx = \left[ \frac{1}{2 \ln 14} \right] \left[ 14^{2x} \right]_0^1 = 37.1345 - 0.1895 = 36.945
\]

b) Applying Composite Simpson’s 1/3 Rule:

\[
I \approx (b - a) \frac{f(x_0) + 4 \sum_{i=1,3,5}^{n-1} f(x_i) + 2 \sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n} \\
a = x_0 = 0, \; b = x_n = x_4 = 1, \; n = 4 \; (given), \; h = \frac{3n}{b - a} = \frac{1 - 0}{4} = 0.25, \; x_1 = 0.25, \; x_2 = 0.5, \; x_3 = 0.75, \; x_4 = 1 \]

\[
I \approx (1 - 0) \frac{f(0) + 4[f(0.25) + f(0.75)] + 2f(0.5) + f(1)}{12}
\]
f(0) = 14^{2(0)} = 1; f(0.25) = 14^{2(0.25)} = 3.7417; f(0.5) = 14^{2(0.5)} = 14; f(0.75) = 14^{2(0.75)} = 52.3832; f(1) = 14^{2(1)} = 196

\[ I \cong (1 - 0) \frac{1 + 4[3.7417 + 52.3832] + 2(14) + 196}{12} = 37.458 \]

e) \((\varepsilon_t, \%)\)

\[ \varepsilon_t (\%) = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 = \left| \frac{36.945 - 37.458}{36.945} \right| \times 100 = 1.39 \%
\]

< 5% correct with at least 1 significant digit

5. a) 

\[ -2 + \sqrt{7} = \sqrt{(-2)^2 + (7)^2} \tan^{-1} \frac{7}{-2} = \sqrt{53} \angle 105.95^\circ \text{ (2nd Quadrant angle)} \]

Applying De Moivre’s theorem:

\[ (-2 + \sqrt{7})^4 = \left[ \sqrt{53} \angle 105.95^\circ \right]^4 = (\sqrt{53})^4 \angle 4(105.95^\circ) = 2809 \angle 423.8^\circ \]

By subtracting 360° (clockwise), equivalent angle will be 63.8° (1st Quadrant angle). The final result will be given in polar form as follows:

\[ (-2 + \sqrt{7})^4 = 2809 \angle 63.8^\circ \text{ (Polar form)} \]

For rectangular form:

\[ r \angle \theta = r(\cos \theta + j \sin \theta) \]

\[ 2809 \angle 63.8^\circ = 2809[\cos(63.8^\circ) + j \sin(63.8^\circ)] = 1240 + j2520 \]

b) Determine the moduli and arguments of the complex roots:

\[ (-2 + \sqrt{7})^{1/4} = (x)^{1/4} = \frac{\sqrt[4]{53}}{4}(\cos \theta + j \sin \theta) \text{ (complex fourth roots)} \]

The roots are symmetrically displaced from one another \(\frac{360^\circ}{4} = 90^\circ\) apart round an Argand diagram as follows:

\[ (-2 + \sqrt{7}) = \sqrt{(-2)^2 + (1)^2} \tan^{-1} \frac{1}{-2} = \sqrt{5} \angle 153.44^\circ \text{ (2nd Quadrant angle)} \]

Applying De Moivre’s theorem:

\[ (-2 + \sqrt{7})^{1/4} = \left[ \sqrt{5} \angle 153.44^\circ \right]^{\frac{1}{4}} = (5)^{\frac{1}{4}} \angle \left( \frac{1}{4} \right)(153.44^\circ) \]

\[ = 1.223 \angle 38.36^\circ \text{ (1st root in 1st Quadrant)}, \]

\[ 1.223 \angle (38.36^\circ + 90^\circ) = 1.223 \angle 128.36^\circ \text{ (2nd root in 2nd Quadrant)} \]

\[ 1.223 \angle (128.36^\circ + 90^\circ) = 1.223 \angle 218.36^\circ \text{ (3rd root in 3rd Quadrant)} \]

3rd root can be given equivalently as follows:

\[ 1.223 \angle 218.36^\circ = 1.223 \angle (-180^\circ + 38.36^\circ) = 1.223 \angle -141.64^\circ \text{ (3rd root in 3rd Quadrant)} \]

\[ 1.223 \angle (218.36^\circ + 90^\circ) = 1.223 \angle 308.36^\circ \text{ (4th root in 4th Quadrant)} \]

4th root can be given equivalently as follows:

\[ 1.223 \angle 308.36^\circ = 1.223 \angle (-141.64^\circ + 90^\circ) = 1.223 \angle -51.64^\circ \text{ (4th root in 4th Quadrant)} \]