1. (25 p) Thermistors are used to measure temperature of bodies. Thermistors are based on materials’ change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes the following observations on a thermistor,

<table>
<thead>
<tr>
<th>R, Ohm</th>
<th>1101.0</th>
<th>911.3</th>
<th>636.0</th>
<th>451.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>T, °C</td>
<td>25.113</td>
<td>30.131</td>
<td>40.120</td>
<td>50.128</td>
</tr>
</tbody>
</table>

Determine the temperature corresponding to 754.8 ohms using Newton’s Divided Difference method and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

2. (25 p) The upward velocity of a rocket is given at three different times on the following table:

<table>
<thead>
<tr>
<th>Time (t), s</th>
<th>5</th>
<th>8</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity (v), m/s</td>
<td>106.8</td>
<td>177.2</td>
<td>279.2</td>
</tr>
</tbody>
</table>

The velocity data is approximated by a polynomial as

\[ v(t) = at^2 + bt + c, \quad 5 \leq t \leq 12. \]

Set up the equations in matrix form to find the coefficients \( a, b, c \) of the velocity profile by using LU Factorization method.

3. (25 p) The following system of equations is designed to determine concentrations (the \( c \)'s in g/m^3) in a series of coupled reactors as a function of the amount of mass input to each reactor (the right hand sides in g/day):

\[
\begin{align*}
15c_1 - 3c_2 - c_3 &= 3800 \\
-3c_1 + 18c_2 - 6c_3 &= 1200 \\
-4c_1 - c_2 + 12c_3 &= 2350
\end{align*}
\]

Solve this problem by using Gauss-Seidel method to the maximum approximate error \( \varepsilon_a \) falls below 10%.

4. (25 p) You buy a 25 000 $ piece of equipment for nothing down at 5 500 $ per year for 6 years. What interest rate are you paying? Use bisection method with initial guesses of \( i_l = 0.05 \) and \( i_u = 0.10 \), and iterate until the approximate error falls below 5%. The formula relating present worth \( P \), annual payments \( A \), number of years \( n \), and interest rate \( i \) is

\[ A = P \cdot \frac{i(1+i)^n}{(1+i)^n-1} \]

Recall for General form of Newton’s Interpolating Polynomials:

\[
\begin{align*}
&f_{n-1}(x) = b_1 + b_2(x-x_1) + \ldots + b_n(x-x_1)(x-x_2)\ldots(x-x_{n-1}) \\
&b_1 = f(x_1); \quad b_2 = f[x_2,x_1]; \quad b_3 = f[x_3,x_2,x_1]; \ldots; \quad b_{n-1} = f[x_{n-1},x_{n-2},\ldots,x_1]; \quad b_n = f[x_n,x_{n-1},\ldots,x_2,x_1] \\
&f[x_i,x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}; \quad f[x_i,x_j,x_k] = \frac{f[x_i,x_j] - f[x_j,x_k]}{x_i - x_k} \\
&f[x_n,x_{n-1},\ldots,x_2,x_1] = \frac{f[x_n,x_{n-1},\ldots,x_2] - f[x_{n-1},x_{n-2},\ldots,x_1]}{x_n - x_1}
\end{align*}
\]
1) From: http://numericalmethods.eng.usf.edu/

For second order polynomial interpolation (also called quadratic interpolation), we choose the temperature given by

\[ T(R) = b_1 + b_2(R - R_1) + b_3(R - R_1)(R - R_2) \]

Since we want to find the temperature at \( R = 754.8 \), we need to choose three data points that are closest to \( R = 754.8 \), and also bracket \( R = 754.8 \). These three points are \( R_1 = 911.3, R_2 = 636.0, \) and \( R_3 = 451.1 \).

\[
\begin{align*}
R_1 &= 911.3, T(R_1) = 30.131 \\
R_2 &= 636.0, T(R_2) = 40.120 \\
R_3 &= 451.1, T(R_3) = 50.128
\end{align*}
\]

then

\[
\begin{align*}
b_1 &= T(R_1) \\
&= 30.131 \\
b_2 &= \frac{T(R_2) - T(R_1)}{R_2 - R_1} \\
&= \frac{40.120 - 30.131}{636.0 - 911.3} \\
&= -0.036284 \\
b_3 &= \frac{(R_3 - R_2)(T(R_1) - T(R_2)) - (R_3 - R_1)(T(R_1) - T(R_2))}{R_3 - R_1} \\
&= \frac{(451.1 - 636.0)(30.131 - 40.120) - (451.1 - 911.3)(30.131 - 40.120)}{451.1 - 911.3} \\
&= -0.054127 + 0.036284 \\
&= -0.017843 \\
&= 3.8771 \times 10^{-5}
\end{align*}
\]

then

\[
\begin{align*}
T(R) &= b_1 + b_2(R - R_1) + b_3(R - R_1)(R - R_2) \\
&= 30.131 - 0.036284(R - 911.3) + 3.8771 \times 10^{-5} (R - 911.3)(R - 636.0), 451.1 \leq R \leq 911.3
\end{align*}
\]

At \( R = 754.8 \),

\[
\begin{align*}
T(754.8) &= 30.131 - 0.036284(754.8 - 911.3) + 3.8771 \times 10^{-5} (754.8 - 911.3)(754.8 - 636.0) \\
&= 35.089°C
\end{align*}
\]

The absolute relative approximate error \( |\varepsilon_a| \) obtained between the results from the first and second order polynomial is

\[
|\varepsilon_a| = \left| \frac{35.089 - 35.089}{35.089} \right| \times 100
\]
If we expand,
\[ T(R) = 30.131 - 0.036284(R - 911.3) + 3.8771 \times 10^{-5} (R - 911.3)(R - 636.0), \quad 451.1 \leq R \leq 911.3 \]
we get
\[ T(R) = 85.668 - 0.09627R + 3.8771 \times 10^{-5} R^2, \quad 451.1 \leq R \leq 911.3 \]

2. From: http://numericalmethods.eng.usf.edu/

The polynomial is going through three data points \((t_1, v_1), (t_2, v_2),\) and \((t_3, v_3)\) where from the table given:
\[ t_1 = 5, v_1 = 106.8 \]
\[ t_2 = 8, v_2 = 177.2 \]
\[ t_3 = 12, v_3 = 279.2 \]

Requiring that \(v(t) = at^2 + bt + c\) passes through the three data points gives
\[ v(t_1) = v_1 = at_1^2 + bt_1 + c \]
\[ v(t_2) = v_2 = at_2^2 + bt_2 + c \]
\[ v(t_3) = v_3 = at_3^2 + bt_3 + c \]

Substituting the data \((t_1, v_1), (t_2, v_2), (t_3, v_3)\) gives
\[ a(5^2) + b(5) + c = 106.8 \]
\[ a(8^2) + b(8) + c = 177.2 \]
\[ a(12^2) + b(12) + c = 279.2 \]

or
\[ 25a + 5b + c = 106.8 \]
\[ 64a + 8b + c = 177.2 \]
\[ 144a + 12b + c = 279.2 \]

This set of equations can be rewritten in the matrix form as
\[
\begin{bmatrix}
25a + 5b + c \\
64a + 8b + c \\
144a + 12b + c
\end{bmatrix} = 
\begin{bmatrix}
106.8 \\
177.2 \\
279.2
\end{bmatrix}
\]

The above equation can be written as a linear combination as follows
\[
a \begin{bmatrix}
25 \\
64 \\
144
\end{bmatrix} + b \begin{bmatrix}
5 \\
8 \\
12
\end{bmatrix} + c \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} = 
\begin{bmatrix}
106.8 \\
177.2 \\
279.2
\end{bmatrix}
\]

and further using matrix multiplications gives
The solution of the above three simultaneous linear equations by using LU Factorization method will give the value of $a, b, \ c$.

Recall that

$$
[A] = [L][U] = \begin{bmatrix}
1 & 0 & 0 \\
\ell_{21} & 1 & 0 \\
\ell_{31} & \ell_{32} & 1 \\
\end{bmatrix}
\begin{bmatrix}
u_{11} & u_{12} & u_{13} \\
u_{22} & u_{22} & u_{23} \\
u_{33} & & \\
\end{bmatrix}
$$

$$
[A][x] = [b] \quad \text{(RHS vector)}
$$

and if

$$
[A] = [L][U]
$$

then first solving

$$
[L][d] = [b]
$$

and then

$$
[U][x] = [d]
$$

gives the solution vector $[x]$.

The $[U]$ matrix is the same as found at the end of the forward elimination of Naïve Gauss elimination method, that is

**Forward Elimination of Unknowns:** Since there are three equations, there will be two steps of forward elimination of unknowns.

$$
\begin{bmatrix}
25 & 5 & 1 \\
64 & 8 & 1 \\
144 & 12 & 1 \\
\end{bmatrix}
$$

**First step:** Divide Row 1 by 25 and then multiply it by 64 and subtract the results from Row 2

$$
\text{Row2} - \left( \frac{\text{Row1}}{25} \right) \times 64 = \begin{bmatrix}
25 & 5 & 1 \\
0 & -4.8 & -1.56 \\
144 & 12 & 1 \\
\end{bmatrix}
$$

Divide Row 1 by 25 and then multiply it by 144 and subtract the results from Row 3

$$
\text{Row3} - \left( \frac{\text{Row1}}{25} \right) \times 144 = \begin{bmatrix}
25 & 5 & 1 \\
0 & -4.8 & -1.56 \\
0 & -16.8 & -4.76 \\
\end{bmatrix}
$$

**Second step:** We now divide Row 2 by -4.8 and then multiply by -16.8 and subtract the results from Row 3
Row3 – \( \begin{bmatrix} \text{Row2} \\ -4.8 \end{bmatrix} \times (-16.8) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \)

\[ U \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \]

To find \( \ell_{21} \) and \( \ell_{31} \), what multiplier was used to make the \( a_{21} \) and \( a_{31} \) elements zero in the first step of forward elimination of Naïve Gauss Elimination Method. It was

\[ \ell_{21} = \frac{64}{25} = 2.56 \]

\[ \ell_{31} = \frac{144}{25} = 5.76 \]

To find \( \ell_{32} \), what multiplier was used to make \( a_{32} \) element zero. Remember \( a_{32} \) element was made zero in the second step of forward elimination. The \([A]\) matrix at the beginning of the second step of forward elimination was

\[
\begin{bmatrix}
25 & 5 & 1 \\
0 & -4.8 & -1.56 \\
0 & -16.8 & -4.76 \\
\end{bmatrix}
\]

So

\[ \ell_{32} = \frac{-16.8}{-4.8} = 3.5 \]

Hence

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
2.56 & 1 & 0 \\
5.76 & 3.5 & 1 \\
\end{bmatrix}
\]

Confirm

\[
[L][U] = [A].
\]

\[
[25 & 5 & 1] \\
[64 & 8 & 1] \\
[144 & 12 & 1] \\
\]
\[ A = LU = \begin{bmatrix} 1 & 0 & 0 & 25 & 5 & 1 \\ 2.56 & 1 & 0 & 0 & -4.8 & -1.56 \\ 5.76 & 3.5 & 1 & 0 & 0 & 0.7 \end{bmatrix} \]

2nd step: Forward substitution
\[
[Lu] \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}
\]

to give
\[ d_1 = 106.8 \]
\[ 2.56d_1 + d_2 = 177.2 \]
\[ 5.76d_1 + 3.5d_2 + d_3 = 279.2 \]

Forward substitution starting from the first equation gives
\[ d_1 = 106.8 \]
\[ d_2 = 177.2 - 2.56d_1 \]
\[ = 177.2 - 2.56(106.8) \]
\[ d_1 = 106.8 \]
\[ d_2 = 177.2 - 2.56d_1 \]
\[ = 177.2 - 2.56(106.8) \]
\[ = -96.208 \]
\[ d_3 = 279.2 - 5.76d_1 - 3.5d_2 \]
\[ = 279.2 - 5.76(106.8) - 3.5(-96.208) \]
\[ = 0.76 \]

Hence
\[ \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix} \]

This matrix is same as the right hand side obtained at the end of the forward elimination steps of Naïve Gauss elimination method. Is this a coincidence?

3rd step: Back substitution
\[
[U] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [d]
\]
\[
\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}
\]
\[ 25a + 5b + c = 106.8 \]
\[-4.8b - 1.56c = -96.208\]

\[0.7c = 0.76\]

From the third equation

\[0.7c = 0.76\]

\[c = \frac{0.76}{0.7} = 1.08571\]

Substituting the value of \(c\) in the second equation,

\[-4.8b - 1.56c = -96.208\]

\[b = \frac{-96.208 + 1.56c}{-4.8}\]

\[= \frac{-96.208 + 1.56(1.08571)}{-4.8} = 19.6905\]

Substituting the value of \(b\) and \(c\) in the first equation,

\[25a + 5b + c = 106.8\]

\[a = \frac{106.8 - 5b - c}{25}\]

\[= \frac{106.8 - 5(19.6905) - 1.08571}{25} = 0.290472\]

The solution vector is

\[
\begin{bmatrix}
  a \\
  b \\
  c
\end{bmatrix}
= \begin{bmatrix}
  0.2905 \\
  19.69 \\
  1.086
\end{bmatrix}
\]

3. From Textbook/Chapra Pr. 12.4

The first iteration can be implemented with the assumption of initial guesses of unknowns as zero:

\[c_1 = \frac{3800 + 3c_1 + c_2}{15} = \frac{3800 + 3(0) + 0}{15} = 253.333\]

\[c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(253.333) + 6(0)}{18} = 108.889\]

\[c_3 = \frac{2350 + 4c_1 + c_2}{12} = \frac{2350 + 4(253.333) + 108.889}{12} = 289.3519\]

Second Iteration:
\[ c_1 = \frac{3800 + 3(108.8889) + 289.3519}{15} = 294.4012 \]
\[ c_2 = \frac{1200 + 3(294.4012) + 6(289.3519)}{18} = 212.1842 \]
\[ c_3 = \frac{2350 + 4(294.4012) + 212.1842}{12} = 311.6491 \]

The absolute approximate error estimates can be computed as follows:

\[ \varepsilon_{a,1} = \frac{|294.4012 - 253.3333|}{294.4012} \times 100\% = 13.95\% \]
\[ \varepsilon_{a,2} = \frac{|212.1842 - 289.3519|}{212.1842} \times 100\% = 48.68\% \]
\[ \varepsilon_{a,3} = \frac{|311.6491 - 289.3519|}{311.6491} \times 100\% = 7.15\% \]

The remainder of the calculations can be summarized as follows:

<table>
<thead>
<tr>
<th>iteration</th>
<th>unknown</th>
<th>value</th>
<th>( \varepsilon_a(%) )</th>
<th>Maximum ( \varepsilon_a(%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( c_1 )</td>
<td>253.3333</td>
<td>100.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c_2 )</td>
<td>108.8889</td>
<td>100.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c_3 )</td>
<td>289.3519</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>2</td>
<td>( c_1 )</td>
<td>294.4012</td>
<td>13.95</td>
<td>48.68</td>
</tr>
<tr>
<td></td>
<td>( c_2 )</td>
<td>212.1842</td>
<td>48.68</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c_3 )</td>
<td>311.6491</td>
<td>7.15</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( c_1 )</td>
<td>316.5468</td>
<td>7.00</td>
<td>7.00&lt;10</td>
</tr>
<tr>
<td></td>
<td>( c_2 )</td>
<td>223.3075</td>
<td>4.98</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c_3 )</td>
<td>319.9579</td>
<td>2.60</td>
<td></td>
</tr>
</tbody>
</table>

We arrive at the result \( c_1=316.5468, c_2=223.3075, c_3=319.9579 \) after 3 iterations with \( \varepsilon_a(10\%) \).

4. From Textbook/Chapra Pr. 5.14
The solution can be formulated as
\[ f(i) = 25000 \frac{i(1+i)^6}{(1+i)^6 - 1} - 5500 = 0; i_L = 0.05; i_u = 0.10 \]

Check: \( f(i_L)f(i_u) = f(0.05)f(0.10) = \left[ 25000 \frac{0.05(1+0.05)^6}{(1+0.05)^6 - 1} - 5500 \right] \left[ 25000 \frac{0.10(1+0.10)^6}{(1+0.10)^6 - 1} - 5500 \right] = (-574.5633)(240.1845)0 \)

(function changes sign: at least one root between initial lower and upper limits)

**Iteration 1:** Bracket \([0.05, 0.10]\) as given
\[ i_r = \frac{i_L + i_u}{2} = \frac{0.05 + 0.10}{2} = 0.075 \]

\[ f(i_L)f(i_r) = f(0.05)f(0.075) = (-574.56) \left[ 25000 \frac{0.075(1+0.075)^6}{(1+0.075)^6 - 1} - 5500 \right] = (-574.56)(-173.88)0 \]

(the root can not be between 0.05 and 0.075; new bracket will be \( i_L = 0.075; i_u = 0.10 \) )

**Iteration 2:** New Bracket \([0.075, 0.10]\)
\[ i_r = \frac{i_i + i_u}{2} = \frac{0.075 + 0.10}{2} = 0.0875 \]

\[ f(i_r)f(i_u) = f(0.075)f(0.0875) = (-173.88)\left[ 25000 \frac{0.0875(1 + 0.0875)^6}{(1 + 0.0875)^6 - 1} - 5500 \right] = (-173.88)(31.52)0 \]

(function changes sign, the root has to be between 0.075 and 0.0875) \[ i_i = 0.075; i_u = 0.0875 \]

**Iteration 3:** New Bracket [0.075, 0.0875]

\[ i_r = \frac{0.075 + 0.0875}{2} = 0.08125 \]

\[ f(0.075)f(0.08125) = (-173.88)\left[ 25000 \frac{0.08125(1 + 0.08125)^6}{(1 + 0.08125)^6 - 1} - 5500 \right] = (-173.88)(-71.59)0 \]

(the root can not be between 0.075 and 0.08125; new bracket will be \[ i_i = 0.08125; i_u = 0.0875 \]

**Iteration 4:** New Bracket [0.08125, 0.0875]

\[ i_r = \frac{0.08125 + 0.0875}{2} = 0.08438 \]

\[ f(0.08125)f(0.08438) = (-71.59)\left[ 25000 \frac{0.08438(1 + 0.08438)^6}{(1 + 0.08438)^6 - 1} - 5500 \right] = (-71.59)(-20.06)0 \]

(the root can not be between 0.08125 and 0.08438; new bracket will be \[ i_i = 0.08438; i_u = 0.0875 \]

\[ \epsilon_a = \left| \frac{i_r^{\text{new}} - i_r^{\text{old}}}{i_r^{\text{new}}} \right| \times 100\% = \left| \frac{0.08438 - 0.08125}{0.08438} \right| \times 100\% = 3.71\%(5\%) \]

The iteration can be terminated with an estimate of root (interest rate) as \[ i = 0.08438 \text{ (8.4%)}. \]

**For lower approximate errors, iteration can be continued as follows:**

**Iteration 5:** New Bracket [0.08438, 0.0875]

\[ i_r = \frac{0.08438 + 0.0875}{2} = 0.08594 \]

\[ f(0.08438)f(0.08594) = (-20.06)\left[ 25000 \frac{0.08594(1 + 0.08594)^6}{(1 + 0.08594)^6 - 1} - 5500 \right] = (-20.06)(5.7069)0 \]

(the root has to be between 0.08438 and 0.08594)

\[ \epsilon_a = \left| \frac{i_r^{\text{new}} - i_r^{\text{old}}}{i_r^{\text{new}}} \right| \times 100\% = \left| \frac{0.08594 - 0.08438}{0.08594} \right| \times 100\% = 1.82\% \]

**Iteration 6:** New Bracket [0.08438, 0.08594]

\[ i_r = \frac{0.08438 + 0.08594}{2} = 0.08516 \]

\[ f(0.08438)f(0.08516) = (-20.06)\left[ 25000 \frac{0.08516(1 + 0.08516)^6}{(1 + 0.08516)^6 - 1} - 5500 \right] = (-20.06)(-7.18)0 \]

(the root can not be between 0.08438 and 0.08516, new bracket will be \[ i_i = 0.08516; i_u = 0.08594 \])

\[ \epsilon_a = \left| \frac{i_r^{\text{new}} - i_r^{\text{old}}}{i_r^{\text{new}}} \right| \times 100\% = \left| \frac{0.08516 - 0.08438}{0.08516} \right| \times 100\% = 0.92\% \]

The root will be estimated after 6 iterations with an approximate error of 0.92% as \[ i = 0.08516 \text{ (8.5%)}. \]