1. (25 p) To find the inverse of a number ‘a’, one can use the equation

\[ f(c) = a - \frac{1}{c} = 0 \]

where \( c \) is the inverse of ‘a’.

Use the Newton-Raphson method of finding roots of equations to

a) Find the inverse of 2.5. Conduct three iterations to estimate the root of the above equation.

b) Find the absolute relative approximate error at the end of each iteration, and

c) The number of significant digits at least correct at the end of each iteration

2. (25 p) An investigator has reported the data tabulated below for an experiment to determine the growth rate of bacteria \( k \) (per d), as a function of oxygen concentration \( c \) (mg/L). It is known that such data can be modeled by the following equation:

\[ k = \frac{k_{\text{max}} c^2}{c_s + c^2} \]

where \( c_s \) and \( k_{\text{max}} \) are parameters. Use a transformation to linearize this equation. Then use linear regression to estimate \( c_s \) and \( k_{\text{max}} \) and predict the growth rate at \( c = 2 \) mg/L.

Data:

<table>
<thead>
<tr>
<th>( c )</th>
<th>0.5</th>
<th>0.8</th>
<th>1.5</th>
<th>2.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>1.1</td>
<td>2.4</td>
<td>5.3</td>
<td>7.6</td>
<td>8.9</td>
</tr>
</tbody>
</table>

3. (25 p) The velocity \( v \) (m/s) of air flowing past a flat surface is measured at several distances \( y \) (m) away from the surface. Determine the shear stress \( \tau \) (N/m²) at the surface (\( y = 0 \)),

\[ \tau = \mu \frac{dv}{dy} \]

Assume a value of dynamic viscosity \( \mu = 1.8 \times 10^{-5} \) N.s/m²

<table>
<thead>
<tr>
<th>( y ), m</th>
<th>0</th>
<th>0.002</th>
<th>0.006</th>
<th>0.012</th>
<th>0.018</th>
<th>0.024</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ), m/s</td>
<td>0</td>
<td>0.287</td>
<td>0.899</td>
<td>1.915</td>
<td>3.048</td>
<td>4.299</td>
</tr>
</tbody>
</table>

4. (25 p) Evaluate the following integral:

\[ \int_{-2}^{2} \left(1 - x - 4x^3 + 2x^5\right) dx \]

(a) analytically, (b) single application of the trapezoidal rule, (c) composite trapezoidal rule with \( n = 2 \) and 4, (d) single application of Simpson’s 1/3 rule, (e) Simpson’s 3/8 rule. For each of the numerical estimates (b) through (e), determine the percent relative error based on (a).
**FORMULA SHEET**

**Interpolation:**

Newton Interpolating Polynomial:

\[ f_{n-1}(x) = b_1 + b_2 (x - x_1) + \ldots + b_n (x - x_1)(x - x_2)\ldots(x - x_{n-1}) \]

\[ b_i = f(x_i) \quad b_2 = f(x_2, x_1); \quad b_3 = f(x_3, x_2, x_1); \ldots; \quad b_{n-1} = f(x_{n-1}, x_2, \ldots, x_1); \quad b_n = f(x_n, x_{n-1}, \ldots, x_2, x_1) \]

\[ f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}; \quad f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k} \]

\[ f[x_n, x_{n-1}, \ldots, x_2, x_1] = f[x_n, x_{n-1}, \ldots, x_2] - f[x_{n-1}, x_2, \ldots, x_1] \]

\[ x_i \text{ } x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

**Lagrange Interpolating Polynomial:**

\[ f_{n-1}(x) = \sum_{i=1}^{n} L_i(x) f(x_i); L_i(x) = \prod_{j=1, j \neq i}^{n} \frac{x - x_j}{x_i - x_j} \]

Regression: \[ y = a_0 + a_1 x \]

\[ a_0 = y - \frac{\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2} \]

**Newton-Raphson formula:**

\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

**Integration:**

\[ I \approx (b-a) \frac{f(x_0) + f(x_n)}{2}; \text{ single Trapezoidal Rule} \]

\[ I \approx \frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}; \text{ composite Trapezoidal Rule} \]

\[ I \approx \frac{h}{3} \left[ f(x_0) + 4f(x_1) + f(x_2) \right] = (b-a) \frac{f(x_0) + 4f(x_1) + f(x_2)}{6}; \text{ single Simpson's 1/3 Rule} \]

\[ I \approx \frac{h}{3} \left[ f(x_0) + 4\sum_{i=1,3,5} f(x_i) + 2\sum_{j=2,4,6} f(x_j) + f(x_n) \right]; \text{ composite Simpson's 1/3 Rule} \]

\[ I \approx \frac{3h}{8} \left[ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right] = (b-a) \frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}; \text{ single Simpson's 3/8 Rule} \]

\[ I \approx \frac{3h}{8} \left[ f(x_0) + 3\sum_{i=1,4,7} f(x_i) + 2\sum_{j=3,6,9} f(x_j) + f(x_n) \right]; \text{ composite Simpson's 3/8 Rule} \]
1. \( f(c) = a - \frac{1}{c} = 0 \)

\[ f'(c) = \frac{1}{c^2} \]

\[ c_{i+1} = c_i - \frac{f(c_i)}{f'(c)} \]

\[ = c_i - \frac{1}{c_i} \]

\[ = c_i \left( 1 - \frac{1}{c_i^2} \right) \]

\[ = c_i - \frac{c_i^2}{c_i} \]

\[ = c_i - c_i^2 + c_i \]

\[ \therefore c_{i+1} = 2c_i - c_i^2 \]

**Iteration #1**

The estimate of the root is \( c_0 = 0.5 \)

\[ c_1 = 2c_0 - c_0^2 a \]

\[ = 2(0.5) - (0.5)^2 (2.5) \]

\[ = 0.375 \]

The absolute relative approximate error, \( |\varepsilon_a| \) at the end of Iteration #1 is

\[ |\varepsilon_a| = \left| \frac{c_1 - c_0}{c_1} \right| \times 100 \]

\[ = \left| \frac{0.375 - 0.5}{0.375} \right| \times 100 \]

\[ = 33.33\% \]

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of less than 5% for one significant digit to be correct in your result.

**Iteration #2**

The estimate of the root is

\[ c_2 = 2c_1 - c_1^2 a \]

\[ = 2(0.375) - (0.375)^2 (2.5) \]

\[ = 0.3984 \]

The absolute relative approximate error, \( |\varepsilon_a| \) at the end of Iteration #2 is

\[ |\varepsilon_a| = \left| \frac{c_2 - c_1}{c_2} \right| \times 100 \]

\[ = \left| \frac{0.3984 - 0.375}{0.3984} \right| \times 100 \]

\[ = 5.8824\% \]

The number of significant digits at least correct is 0.
**Iteration #3**

The estimate of the root is

\[ c_3 = 2c_2 - c_2^2a \]

\[ = 2(0.3984) - (0.3984)^2 (2.5) \]

\[ = 0.3999 \]

The absolute relative approximate error, \(|\varepsilon_a|\) at the end of Iteration #3 is

\[ |\varepsilon_a| = \left| \frac{0.3999 - 0.3984}{0.3999} \right| \times 100 \]

\[ = 0.3891\% \]

Hence the number of significant digits at least correct is given by the largest value of ‘m’ for which

\[ |\varepsilon_a| < \varepsilon_j = 0.5 \times 10^{-m} \]

\[ 0.3891 < 0.5 \times 10^{-m} \]

\[ 0.7782 < 10^{-m} \]

\[ \log(0.7782) < 2 - m \]

\[ m < 2 - \log(0.7782) = 2.1089 \]

So \( m = 2 \)

The number of significant digits at least correct in the estimated root of 0.3999 is 2.

2. The equation can be linearized by inverting it to yield

\[ \frac{1}{k} = \frac{c_s}{k_{\text{max}} c^2} + \frac{1}{k_{\text{max}}} \]

Consequently, a plot of \(1/k\) versus \(1/c\) should yield a straight line with an intercept of \(1/k_{\text{max}}\) and a slope of \(c_s/k_{\text{max}}\)

<table>
<thead>
<tr>
<th>(c, \text{ mg/L})</th>
<th>(k, 1/d)</th>
<th>(1/c^2)</th>
<th>(1/k)</th>
<th>(1/c^2 \times 1/k)</th>
<th>((1/c^2)^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.1</td>
<td>4.000000</td>
<td>0.909091</td>
<td>3.636364</td>
<td>16.000000</td>
</tr>
<tr>
<td>0.8</td>
<td>2.4</td>
<td>1.562500</td>
<td>0.416667</td>
<td>0.651042</td>
<td>2.441406</td>
</tr>
<tr>
<td>1.5</td>
<td>5.3</td>
<td>0.444444</td>
<td>0.188679</td>
<td>0.083857</td>
<td>0.197531</td>
</tr>
<tr>
<td>2.5</td>
<td>7.6</td>
<td>0.160000</td>
<td>0.131579</td>
<td>0.021053</td>
<td>0.025600</td>
</tr>
<tr>
<td>4</td>
<td>8.9</td>
<td>0.062500</td>
<td>0.112360</td>
<td>0.007022</td>
<td>0.003906</td>
</tr>
<tr>
<td><strong>Sum</strong></td>
<td><strong>6.229444</strong></td>
<td><strong>1.758375</strong></td>
<td><strong>4.399338</strong></td>
<td><strong>18.66844</strong></td>
<td></td>
</tr>
</tbody>
</table>

The slope and the intercept can be computed as

\[ a_1 = \frac{5(4.399338) - 6.229444(1.758375)}{5(18.66844) - (6.229444)^2} = 0.202489 \]

\[ a_0 = \frac{1.758375}{5} - \frac{0.202489 \times 6.229444}{5} = 0.099396 \]

Therefore, \(k_{\text{max}} = 1/0.099396 = 10.06074\) and \(c_s = 10.06074(0.202489) = 2.037189\), and the fit is

\[ k = \frac{10.06074c^2}{2.037189 + c^2} \]

The equation can be used to compute
3. The velocity at the surface can be computed by using second order Lagrange polynomial function as

\[
x_0 = 0 \quad f(x_0) = 0
\]
\[
x_1 = 0.002 \quad f(x_1) = 0.287
\]
\[
x_2 = 0.006 \quad f(x_2) = 0.899
\]

The second order Lagrange polynomial passing through \((x_0, y_0), (x_1, y_1), (x_2, y_2)\) is

\[
f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)
\]

Differentiating equation above gives

\[
f_2'(x) = \frac{2x-(x_1+x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2x-(x_0+x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2x-(x_0+x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)
\]

\[
f''(0) = \frac{2(0) - 0.002 - 0.006}{(0-0.002)(0-0.006)} + \frac{2(0) - 0 - 0.006}{(0.002-0)(0.002-0.006)} + \frac{2(0) - 0 - 0.002}{(0.006-0)(0.006-0.002)}
\]

Therefore, the shear stress can be computed as

\[
\tau = 1.8 \times 10^{-5} \frac{N\cdot s}{m^2} \cdot 140.3333 \cdot \frac{1}{s} = 0.00253 \cdot \frac{N}{m^2}
\]

4. (a) Analytical solution:

\[
\int_{-2}^{4} (1 - x - 4x^3 + 2x^5) \, dx = \left[ x - \frac{x^2}{2} - x^4 + \frac{x^6}{3} \right]_{-2}^{4} = 1104
\]

(b) Trapezoidal rule \((n = 1)\):

\[
I = (4 - (-2)) \frac{-29 + 1789}{2} = 5280 \quad \varepsilon_I = \left| \frac{1104 - 5280}{1104} \right| \times 100\% = 378.26\%
\]

(c) Trapezoidal rule \((n = 2)\):

\[
I = (4 - (-2)) \frac{-29 + 2(-2) + 1789}{4} = 2634 \quad \varepsilon_I = 138.59\%
\]

Trapezoidal rule \((n = 4)\):

\[
I = (4 - (-2)) \frac{-29 + 2(1.9375 - 2 + 131.3125) + 1789}{8} = 1516.875 \quad \varepsilon_I = 37.398\%
\]

(d) Simpson’s 1/3 rule \((n = 2)\):
\[ I = (4 - (-2)) \frac{-29 + 4(-2) + 1789}{6} = 1752 \quad \varepsilon_r = 58.7\% \]

(e) Simpson’s 3/8 rule:

\[ I = (4 - (-2)) \frac{-29 + 3(1 + 31) + 1789}{8} = 1392 \quad \varepsilon_r = 26.087\% \]