1. (35 p) You must determine the root of the following easily differentiable function: 
\[ e^{0.5x} = 5 - 5x \]

a) Use the **Bisection** method with an absolute approximate relative percentage error \((\epsilon_a)\) below 2%. Use initial guesses of \(x_l = 0, and \ x_u = 2\)

b) Use **Newton-Raphson** method with an absolute approximate relative percentage error \((\epsilon_a)\) below 0.5%. Use initial guess \(x_i = 0.7\).

2. (35 p) An electronic company produces transistors, resistors, and computer chips. Each transistor requires four units of copper, one unit of zinc, and two units of glass. Each resistor requires three, three, and one units of the three materials, respectively, and each computer chip requires two, one, and three units of these materials respectively. Putting this information into table form, we get:

<table>
<thead>
<tr>
<th>Component</th>
<th>Copper</th>
<th>Zinc</th>
<th>Glass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transistors</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Resistors</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Computer chips</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Supplies of these materials vary from week to week, so the company needs to determine a different production run each week. For example, one week the total amounts of materials available are 960 units of copper, 510 units of zinc, and 610 units of glass. Set up the system of equations modeling the production run and solve these equations for the number of transistors, resistors, and computer chips to be manufactured this week. Use Gauss-Seidel method without relaxation to \(\epsilon_r = 1\%\).

3. (30 p) Rather than using the base-\(e\) exponential model, a common alternative is to use a base-10 model,

\[ y = a \times 10^{b \times x} \]

Use this model given above to fit the following data:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2</th>
<th>2.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>800</td>
<td>975</td>
<td>1500</td>
<td>1950</td>
<td>2900</td>
<td>3600</td>
</tr>
</tbody>
</table>

Recall:

\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

\([A]\{x\} = \{b\}; \ [L]\{U\} = [A]; \ [L]\{d\} = \{b\}; \ [U]\{x\} = \{d\}\]

\[ y = a_0 + a_1 x \]

\[ a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} \]

\[ a_0 = \bar{y} - a_1 \bar{x} \]
1. a) Check function changes sign: at least one root between initial lower and upper limits.

\[ f(x) = e^{0.5x} - 5 + 5x = 0 \]

\[ f(x_l)f(x_u) = f(0)f(2) = (-4.000)(7.718) < 0 \] (function changes sign, the root has to be between 0 and 2)

**Step 1:** Bracket \([0, 2]\) as given

- \[ x_r = \frac{x_l + x_u}{2} = \frac{0 + 2}{2} = 1 \]
- \[ f(x_l) = f(0) = e^{0.5(0)} - 5 + 5(0) = -4.000 \]
- \[ f(x_r) = f(1) = e^{0.5(1)} - 5 + 5(1) = 1.649 \]
- \[ f(x_l)f(x_r) = (-4.000)(1.649) < 0 \] (function changes sign, the root has to be between 0 and 1; change \(x_r\) as upper limit, lower limit will be same).

**Step 2:** New Bracket \([0, 1]\)

- \[ x_r = \frac{x_l + x_u}{2} = \frac{0 + 1}{2} = 0.5 \]
- \[ f(x_l) = f(0) = -4.000 \]
- \[ f(x_r) = f(0.5) = e^{0.5(0.5)} - 5 + 5(0.5) = -1.216 \]
- \[ f(x_l)f(x_r) = (-4.000)(-1.216) > 0 \] (the root cannot be between 0 and 0.5; change \(x_r\) as lower limit, upper limit will be same).

New bracket will be \([0.5, 1]\)

\[ \epsilon_a = \left| \frac{0.5 - 1}{0.5} \right| \times 100 = 100\% \]

**Step 3:** New Bracket \([0.5, 1]\)

- \[ x_r = \frac{x_l + x_u}{2} = \frac{0.5 + 1}{2} = 0.75 \]
- \[ f(x_l) = f(0.5) = -1.216 \]
- \[ f(x_r) = f(0.75) = e^{0.5(0.75)} - 5 + 5(0.75) = 0.205 \]
- \[ f(x_l)f(x_r) = (-1.216)(0.205) < 0 \] (the root has to be between 0.5 and 0.75; new bracket will be \([0.5, 0.75]\))

\[ \epsilon_a = \left| \frac{0.75 - 0.5}{0.75} \right| \times 100 = 33.33\% > 2\% \]

**Step 4:** New Bracket \([0.5, 0.75]\)

- \[ x_r = \frac{x_l + x_u}{2} = \frac{0.5 + 0.75}{2} = 0.625 \]
- \[ f(x_l) = f(0.5) = -1.216 \]
- \[ f(x_r) = f(0.625) = e^{0.5(0.625)} - 5 + 5(0.625) = -0.508 \]
- \[ f(x_l)f(x_r) = (-1.216)(-0.508) > 0 \] (the root can’t be between 0.5 and 0.625; new bracket will be \([0.625, 0.75]\))

\[ \epsilon_a = \left| \frac{0.625 - 0.75}{0.625} \right| \times 100 = 20\% > 2\% \]

**Step 5:** New Bracket \([0.625, 0.75]\)

- \[ x_r = \frac{x_l + x_u}{2} = \frac{0.625 + 0.75}{2} = 0.6875 \]
- \[ f(x_l) = f(0.625) = -0.508 \]
\[
f(x_r) = f(0.6875) = e^{0.5(0.6875)} - 5 + 5(0.6875) = -0.1523
f(x_l)f(x_r) = (-0.508)(-0.1523) > 0
\]

(the root can’t be between 0.625 and 0.6875; new bracket will be [0.6875, 0.71875])

\[
\varepsilon_a = \left| \frac{0.6875 - 0.625}{0.6875} \right| \times 100 = 9.09\% > 2\%
\]

**Step 6:** New Bracket [0.6875, 0.75]

\[
x_r = \frac{x_l + x_u}{2} = \frac{0.6875 + 0.75}{2} = 0.71875
f(x_l) = f(0.6875) = -0.1523
f(x_r) = f(0.71875) = e^{0.5(0.71875)} - 5 + 5(0.71875) = 0.02618
f(x_l)f(x_r) = (-0.1523)(0.02618) < 0
\]

(the root has to be between 0.6875 and 0.71875; new bracket will be [0.6875, 0.71875])

\[
\varepsilon_a = \left| \frac{0.71875 - 0.6875}{0.71875} \right| \times 100 = 4.35\% > 2\%
\]

Thus after 8 iterations, the approximate error falls below 2\% with a result of \( x_r = 0.7109 \) as tabulated:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_l )</th>
<th>( x_u )</th>
<th>( x_r )</th>
<th>( f(x_l) )</th>
<th>( f(x_r) )</th>
<th>( f(x_l)f(x_r) )</th>
<th>( \varepsilon_a, % )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000</td>
<td>2.0000</td>
<td>1.0000</td>
<td>-4.000</td>
<td>1.649</td>
<td>&lt;0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.5000</td>
<td>-4.000</td>
<td>-1.216</td>
<td>&gt;0</td>
<td>100.00</td>
</tr>
<tr>
<td>3</td>
<td>0.5000</td>
<td>1.0000</td>
<td>0.7500</td>
<td>-1.216</td>
<td>0.205</td>
<td>&lt;0</td>
<td>33.33</td>
</tr>
<tr>
<td>4</td>
<td>0.5000</td>
<td>0.7500</td>
<td>0.6250</td>
<td>-1.216</td>
<td>-0.508</td>
<td>&gt;0</td>
<td>20.00</td>
</tr>
<tr>
<td>5</td>
<td>0.6250</td>
<td>0.7500</td>
<td>0.6875</td>
<td>-0.508</td>
<td>-0.1523</td>
<td>&gt;0</td>
<td>9.09</td>
</tr>
<tr>
<td>6</td>
<td>0.6875</td>
<td>0.7500</td>
<td>0.71875</td>
<td>-0.1523</td>
<td>0.02618</td>
<td>&lt;0</td>
<td>4.35</td>
</tr>
<tr>
<td>7</td>
<td>0.6875</td>
<td>0.71875</td>
<td>0.703125</td>
<td>-0.1523</td>
<td>-0.063</td>
<td>&gt;0</td>
<td>2.22</td>
</tr>
<tr>
<td>8</td>
<td>0.703125</td>
<td>0.71875</td>
<td><strong>0.7109</strong></td>
<td>-0.063</td>
<td>-0.01865</td>
<td>&gt;0</td>
<td>1.09&lt;2%</td>
</tr>
</tbody>
</table>

b) The solution can be formulated as:

\[
f(x) = e^{0.5x} - 5 + 5x = 0
f'(x) = 0.5e^{0.5x} + 5
\]

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

Applying Newton-Raphson formula given above gives with the initial guess of \( x_0 = 0.7 \)

**Iteration 1:**

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.7 - \frac{e^{0.5(0.7)} - 5 + 5(0.7)}{0.5e^{0.5(0.7)} + 5} = 0.714175
\]

\[
\varepsilon_a = \left| \frac{0.714175 - 0.7}{0.714175} \right| \times 100 = 1.98\% > 0.5\%
\]

**Iteration 2:**

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.714175 - \frac{e^{0.5(0.714175)} - 5 + 5(0.714175)}{0.5e^{0.5(0.714175)} + 5} = 0.714169
\]
\[ \epsilon_a = \left| \frac{0.714169 - 0.714175}{1.7500} \right| \times 100 = 0.00084\% < 0.5\% \]

Root \( x = 0.714169 \) with 4 significant digits

2. Define the quantity of transistors, resistors, and computer chips as \( x_1, x_2, x_3 \). The system equations can then be defined as

\[
\begin{align*}
4x_1 + 3x_2 + 2x_3 &= 960 \\
x_1 + 3x_2 + x_3 &= 510 \\
2x_1 + x_2 + 3x_3 &= 610
\end{align*}
\]

(The equations should first be rearranged so that they are diagonally dominant, if not)

Each can be solved for the unknown on the diagonal as

\[
\begin{align*}
x_1 &= \frac{960 - 3x_2 - 2x_3}{4} \\
x_2 &= \frac{510 - x_1 - x_3}{3} \\
x_3 &= \frac{610 - 2x_1 - x_2}{3}
\end{align*}
\]

The first iteration can be implemented as with initial values \( x_1, x_2, x_3 = 0 \)

\[
\begin{align*}
x_1 &= \frac{960 - 3(0) - 2(0)}{4} = 240 \\
x_2 &= \frac{510 - 240 - 0}{3} = 90 \\
x_3 &= \frac{610 - 2(240) - 90}{3} = 13.33
\end{align*}
\]

Second iteration:

\[
\begin{align*}
x_1 &= \frac{960 - 3(90) - 2(13.33)}{4} = 165.83 \\
x_2 &= \frac{510 - 165.83 - 13.33}{3} = 110.279 \\
x_3 &= \frac{610 - 2(165.83) - 110.279}{3} = 56.02
\end{align*}
\]

The error estimates can be computed as

\[
\begin{align*}
\epsilon_{a,1} &= \left| \frac{165.83 - 240}{165.83} \right| \times 100 = 44.73\% \\
\epsilon_{a,2} &= \left| \frac{110.279 - 90}{110.279} \right| \times 100 = 18.39\% \\
\epsilon_{a,3} &= \left| \frac{56.02 - 13.33}{56.02} \right| \times 100 = 76.21\%
\end{align*}
\]

Third iteration:

\[
\begin{align*}
x_1 &= \frac{960 - 3(110.279) - 2(56.02)}{4} = 129.28
\end{align*}
\]
The error estimates can be computed as
\[
\varepsilon_{a,1} = \left| \frac{129.28 - 165.83}{129.28} \right| \times 100 = 28.27\%
\]
\[
\varepsilon_{a,2} = \left| \frac{108.23 - 110.279}{108.23} \right| \times 100 = 1.89\%
\]
\[
\varepsilon_{a,3} = \left| \frac{81.07 - 56.02}{81.07} \right| \times 100 = 30.90\%
\]

**Fourth iteration:**
\[
x_1 = \frac{960 - 3(108.23) - 2(81.07)}{4} = 118.29
\]
\[
x_2 = \frac{510 - 118.29 - 81.07}{3} = 103.55
\]
\[
x_3 = \frac{610 - 2(118.29) - 103.55}{3} = 89.96
\]

The error estimates can be computed as
\[
\varepsilon_{a,1} = \left| \frac{118.29 - 129.28}{118.29} \right| \times 100 = 9.29\%
\]
\[
\varepsilon_{a,2} = \left| \frac{103.55 - 108.23}{103.55} \right| \times 100 = 4.52\%
\]
\[
\varepsilon_{a,3} = \left| \frac{89.96 - 81.07}{89.96} \right| \times 100 = 9.88\%
\]

**Fifth iteration:**
\[
x_1 = \frac{960 - 3(103.55) - 2(89.96)}{4} = 117.36
\]
\[
x_2 = \frac{510 - 117.36 - 89.96}{3} = 100.89
\]
\[
x_3 = \frac{610 - 2(117.36) - 100.89}{3} = 91.46
\]

The error estimates can be computed as
\[
\varepsilon_{a,1} = \left| \frac{117.36 - 118.29}{117.36} \right| \times 100 = 0.79 < 1\%
\]
\[
\varepsilon_{a,2} = \left| \frac{100.84 - 103.55}{100.84} \right| \times 100 = 2.6 > 1\%.
\]
Sixth iteration:

\[ \varepsilon_{a,3} = \left| \frac{91.46 - 89.96}{91.46} \right| \times 100 = 1.64 > 1\% \]

\[ x_1 = \frac{960 - 3(100.89) - 2(91.46)}{4} = 118.60 \]
\[ x_2 = \frac{510 - 118.60 - 91.46}{3} = 99.98 \]
\[ x_3 = \frac{610 - 2(118.60) - 99.98}{3} = 90.94 \]

The error estimates can be computed as

\[ \varepsilon_{a,1} = \left| \frac{118.60 - 117.36}{118.60} \right| \times 100 = 1.05 \approx 1\% \]
\[ \varepsilon_{a,2} = \left| \frac{99.98 - 100.89}{99.98} \right| \times 100 = 0.91 < 1\% \]
\[ \varepsilon_{a,3} = \left| \frac{90.94 - 91.46}{90.94} \right| \times 100 = 0.57 < 1\% \]

Thus, after 6 iterations, the maximum error is 1.05% and we arrive at the result:

\( x_1 = 118.60 \) (number of transistors 119)
\( x_2 = 99.98 \) (number of resistors 100)
\( x_3 = 90.94 \) (number of computer chips 91)

3.

\[ y = \alpha_5 \ 10^{\beta_5 x} \]

Transformation to linearize base-10 model will give:

\[ \log_{10} y = \log_{10} \alpha_5 + \beta_5 x \]

Assuming

\[ Z = \log_{10} y \]
\[ a_0 = \log_{10} \alpha_5; \ a_1 = \beta_5 \]
\[ \alpha_5 = 10^{a_0}; \ a_1 = \beta_5 \]

We get

\[ Z = a_0 + a_1 x \]

This is a linear relationship between \( z \) and \( x \).

\[ a_1 = \frac{n \sum^n x_i Z_i - \sum^n x_i \sum^n Z_i}{n \sum^n x_i^2 - (\sum^n x_i)^2} \]
\[ a_0 = \bar{Z} - a_1 \bar{x} \]
Table. Summations of data to calculate parameters of the base-10 exponential model.

<table>
<thead>
<tr>
<th>i</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( z_i = \log_{10} y_i )</th>
<th>( x_i z_i )</th>
<th>( x_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
<td>800</td>
<td>2.9031</td>
<td>1.1612</td>
<td>0.16</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>975</td>
<td>2.9890</td>
<td>2.3912</td>
<td>0.64</td>
</tr>
<tr>
<td>3</td>
<td>1.2</td>
<td>1500</td>
<td>3.1761</td>
<td>3.8113</td>
<td>1.44</td>
</tr>
<tr>
<td>4</td>
<td>1.6</td>
<td>1950</td>
<td>3.2900</td>
<td>5.2640</td>
<td>2.56</td>
</tr>
<tr>
<td>5</td>
<td>2.0</td>
<td>2900</td>
<td>3.4624</td>
<td>6.9248</td>
<td>4.00</td>
</tr>
<tr>
<td>6</td>
<td>2.3</td>
<td>3600</td>
<td>3.5563</td>
<td>8.1795</td>
<td>5.29</td>
</tr>
<tr>
<td>Σ</td>
<td>8.3</td>
<td>19.3769</td>
<td>27.732</td>
<td>14.09</td>
<td>14.09</td>
</tr>
</tbody>
</table>

From the least square equations given above, we have

\[
a_1 = \frac{6(27.732) - (8.3)(19.3769)}{6(14.09) - (8.3)^2} = 0.3555
\]
\[
a_0 = \frac{19.3769}{6} - 0.3555 \frac{8.3}{6} = 2.7377
\]

Since

\[
\alpha_5 = 10^{a_0} = 10^{2.7377} = 546.6382; \quad a_1 = \beta_5 = 0.3555
\]

Fitting equation to the data:

\[
y = \alpha_5 \ 10^{\beta x} = (546.6382)10^{0.3555x}
\]